General Purpose SAT-Solvers for Causal Discovery

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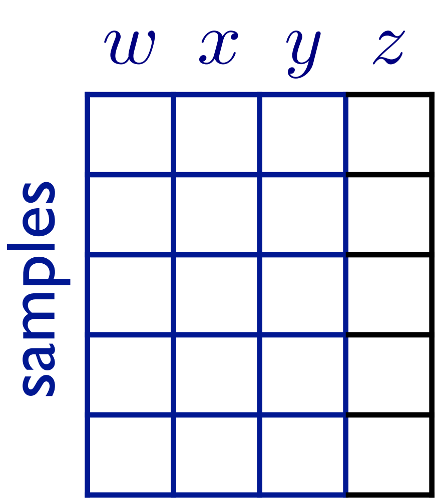
Caltech

[joint work with Antti Hyttinen and Matti Jarvisalo]

CCD 1/21/2016

Causal Discovery

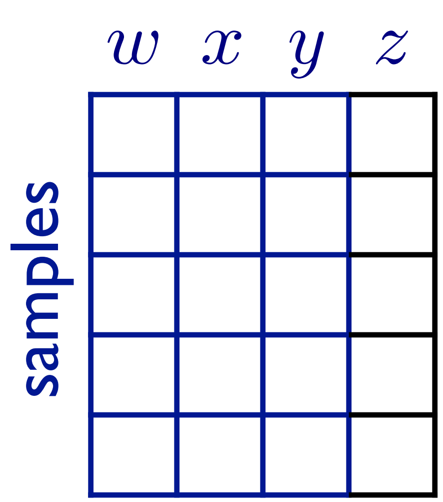
data
sample



Causal Discovery

assumptions, e.g.
- causal Markov
- causal faithfulness
- functional form
- etc.

data sample



inference algorithm

Causal Discovery

data sample

	w	x	y	z
samples				

assumptions, e.g.
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- causal faithfulness
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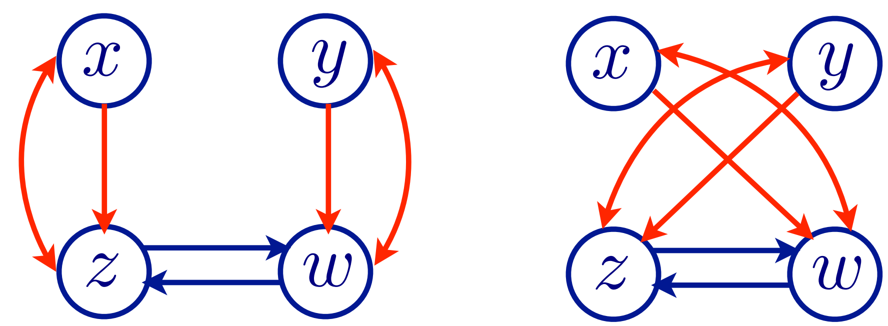
- etc.

inference algorithm

equivalence classes



model specifications

	w	x	y	z
w	0	0	?	a
x	0	0	0	0
y	0	0	0	0
z	b	?	?	0

direct edges

	w	x	y	z
w	0	0	?	?
x	0	0	0	?
y	?	0	0	0
z	?	?	0	0

confounders

General Model Space

assumption / algorithm
Markov
faithfulness
causal sufficiency
acyclicity
parametric assumption

General Model Space

assumption / algorithm	PC / GES	FCI	CCD	LiNGaM	non-linear additive noise
Markov	✓	✓	✓	✓	✓
faithfulness	✓	✓	✓	✗	minimality
causal sufficiency	✓	✗	✓	✓	✓
acyclicity	✓	✓	✗	✓	✓
parametric assumption	✗	✗	✗	linear non-Gaussian	non-linear additive noise

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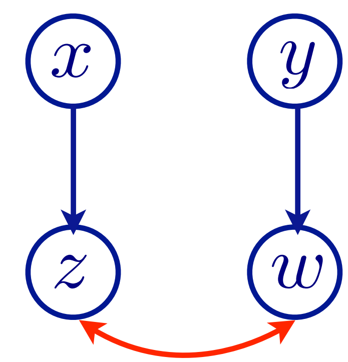
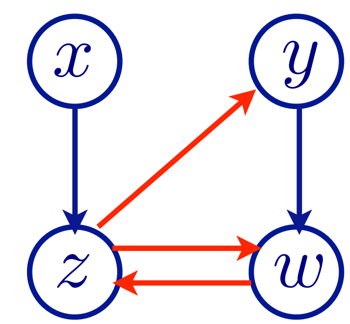
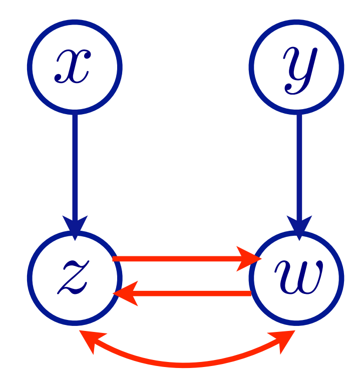


w/ latents



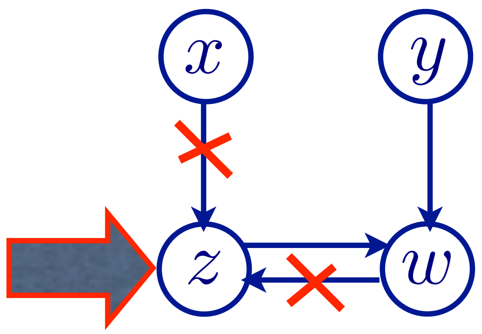
w/ cycles



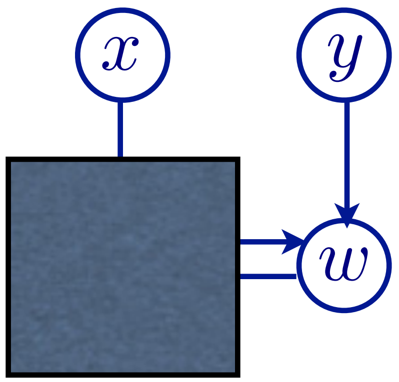
w/ both

Combining Experiment and Observation

experiment



observational study



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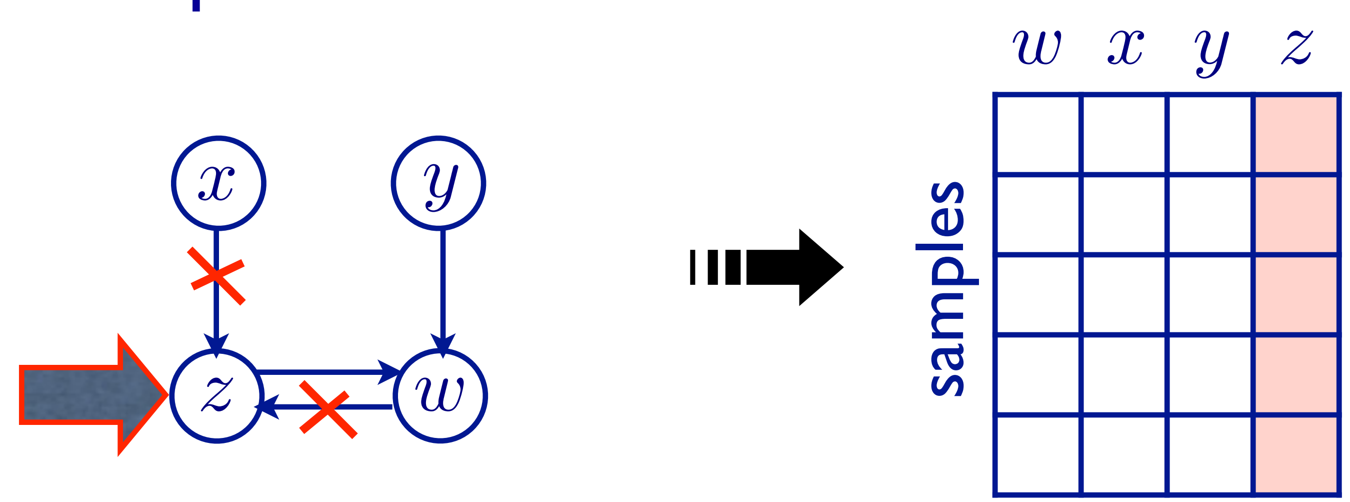

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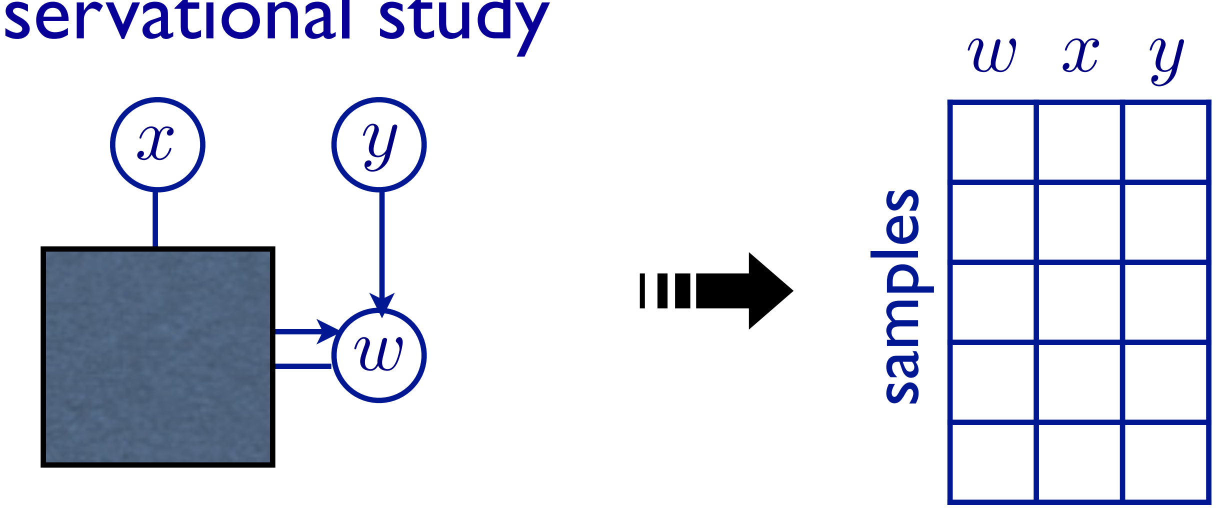
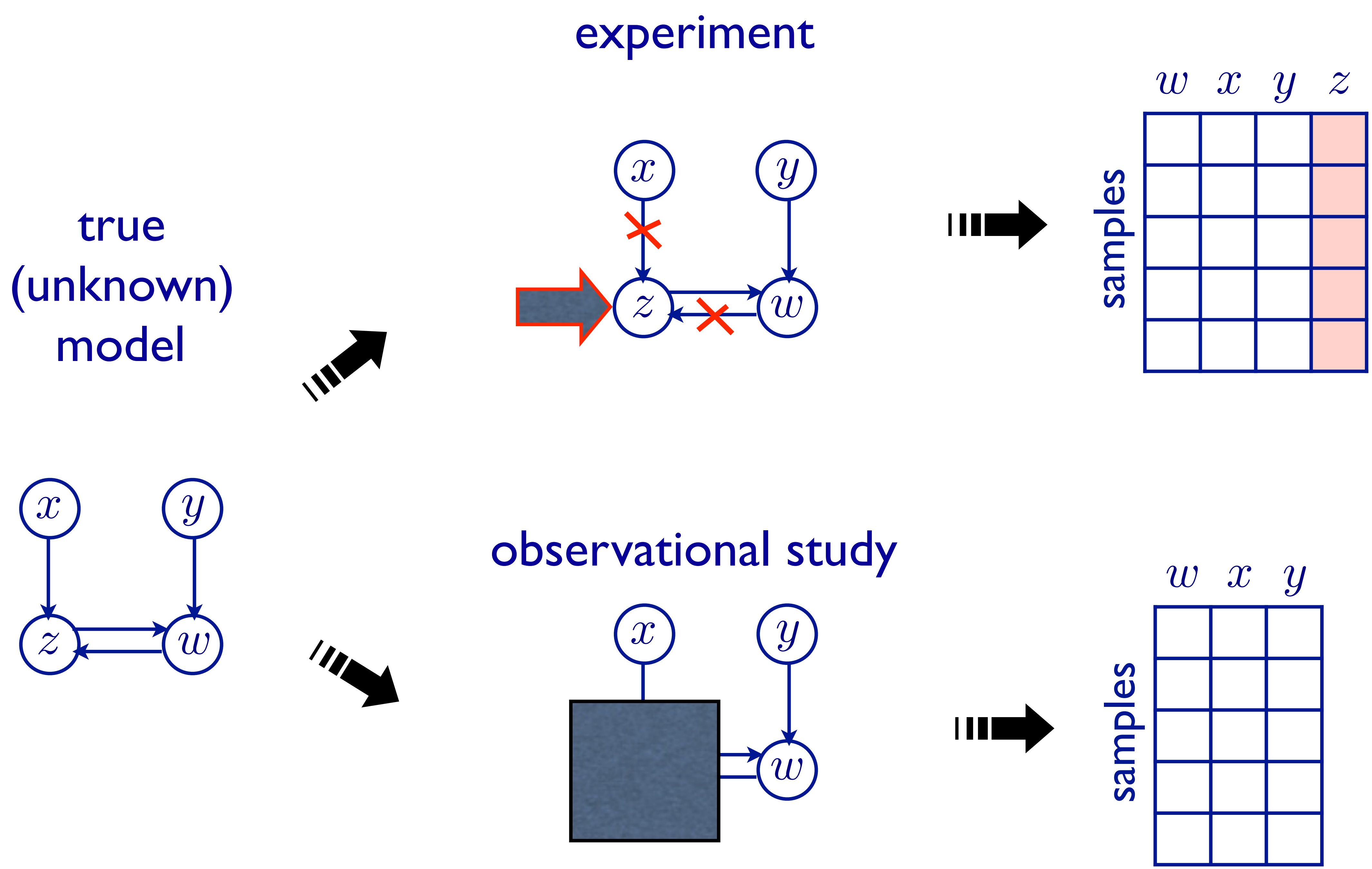
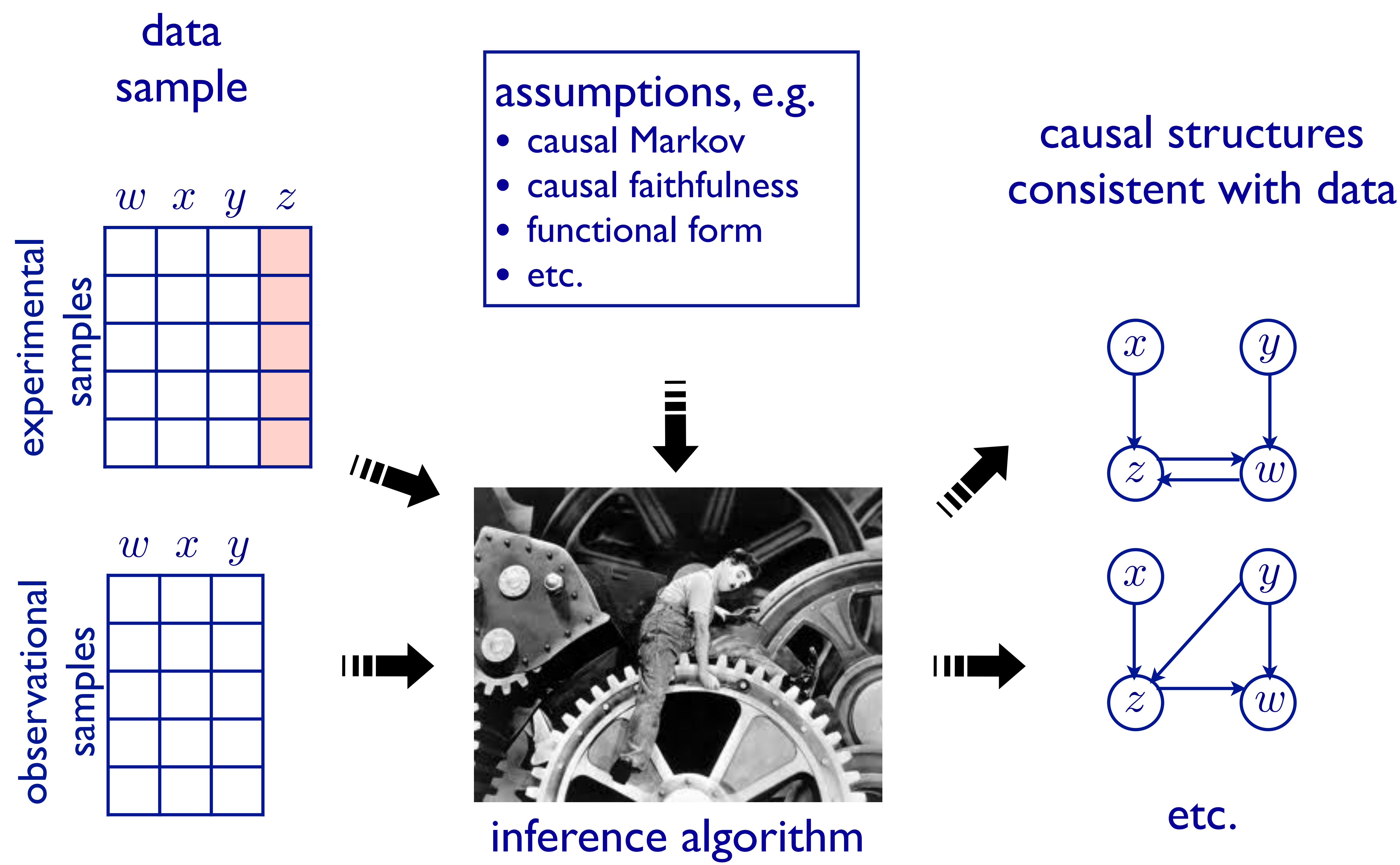


Combining Experiment and Observation



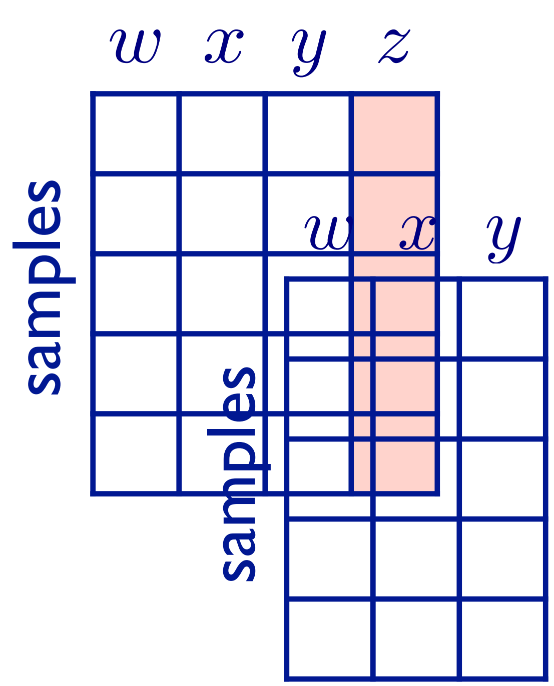
Causal Discovery



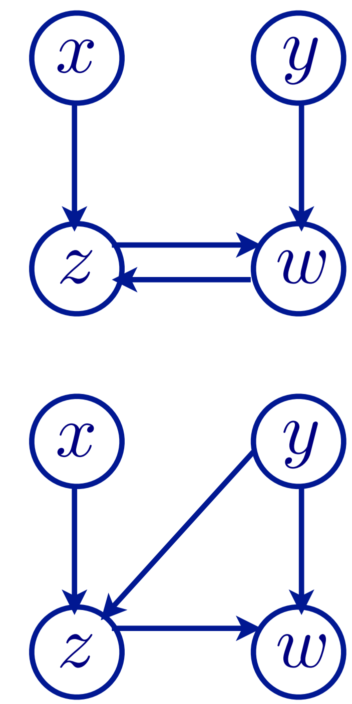
Causal Discovery

assumptions, e.g.
- causal Markov
- causal faithfulness
- functional form
- etc.

causal structures consistent with data



inference algorithm



etc.

Causal Discovery

background knowledge

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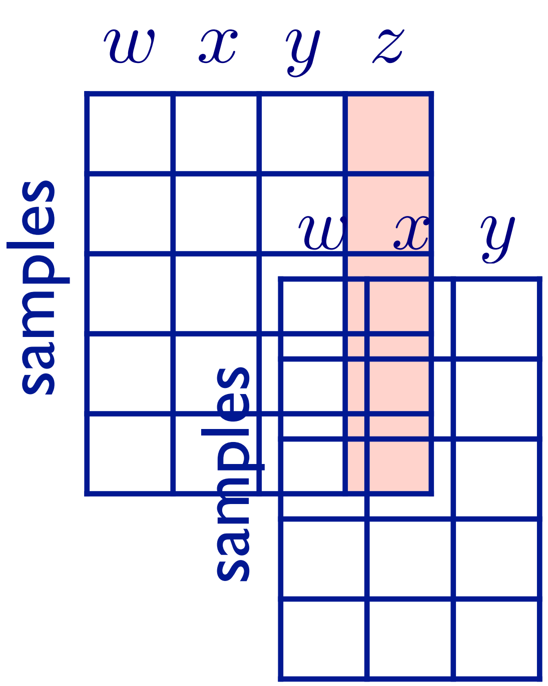

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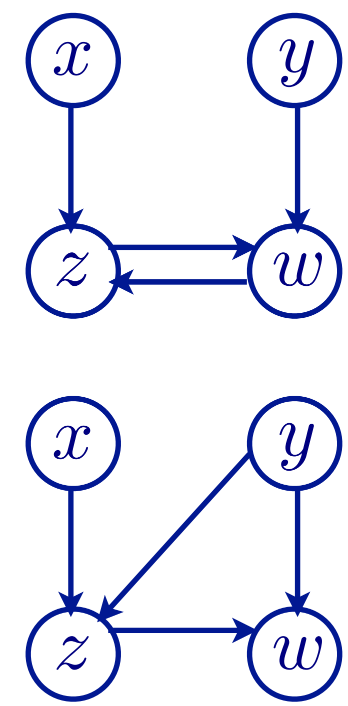

etc.

Causal Discovery

background knowledge

- edge presences/ absences

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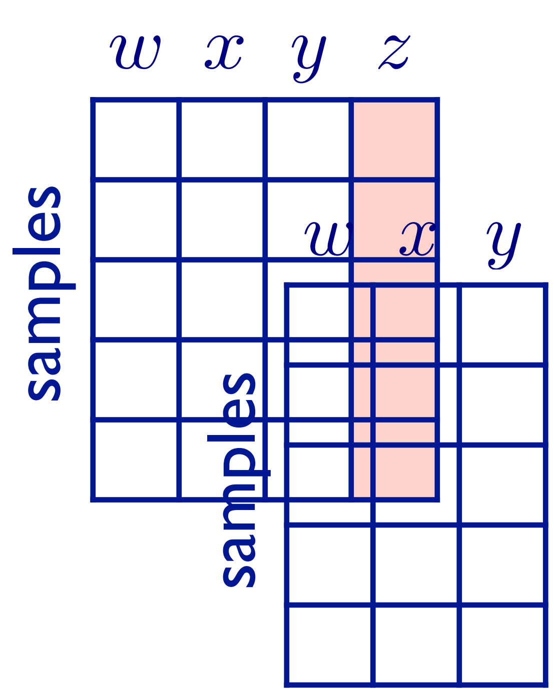

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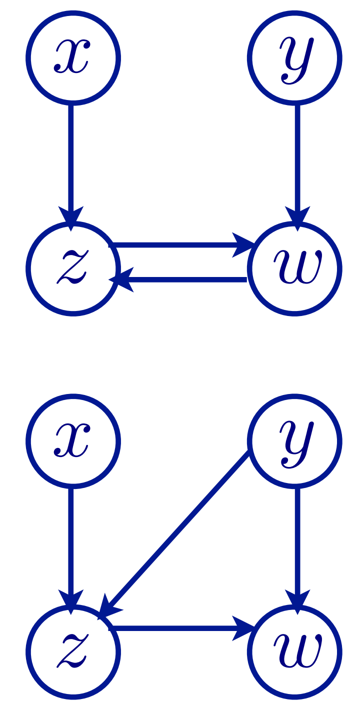
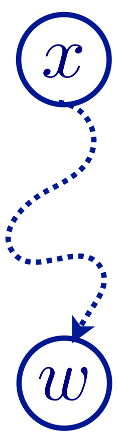


etc.

Causal Discovery

background knowledge



- edge presences/ absences
- pathways

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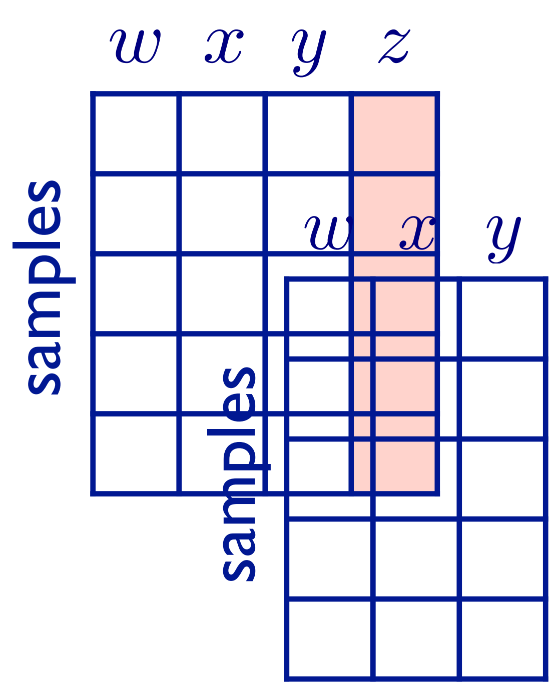
causal structures consistent with data



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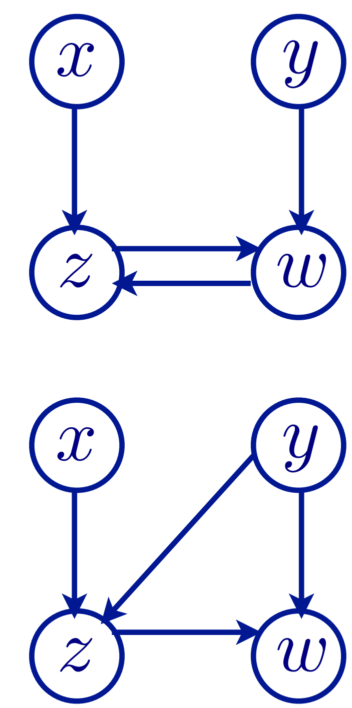

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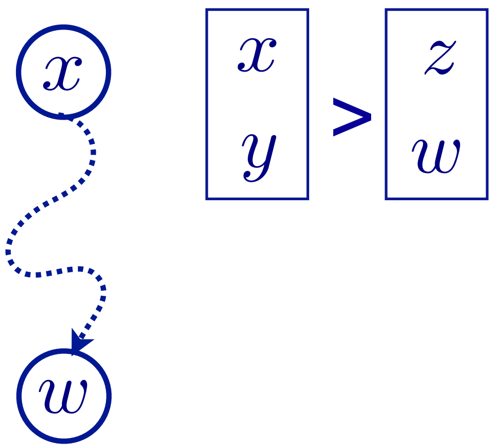
Causal Discovery

background knowledge



- edge presences/ absences
- pathways
- tier orderings

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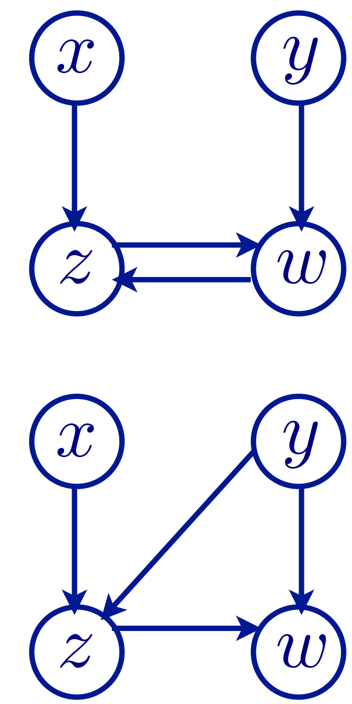
causal structures consistent with data



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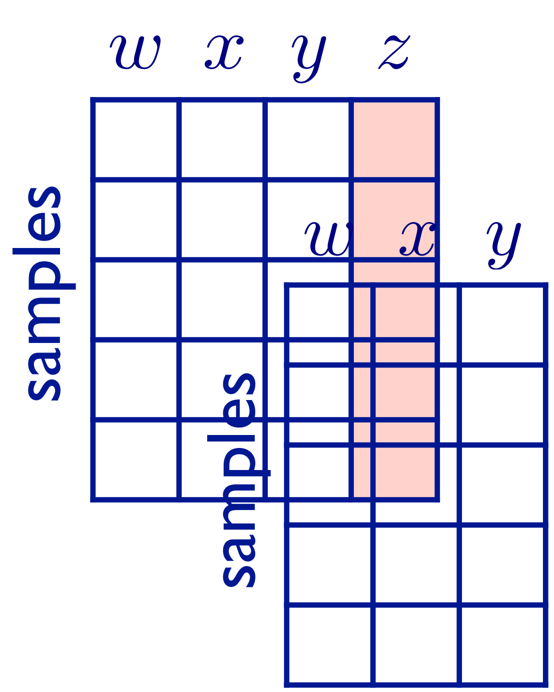
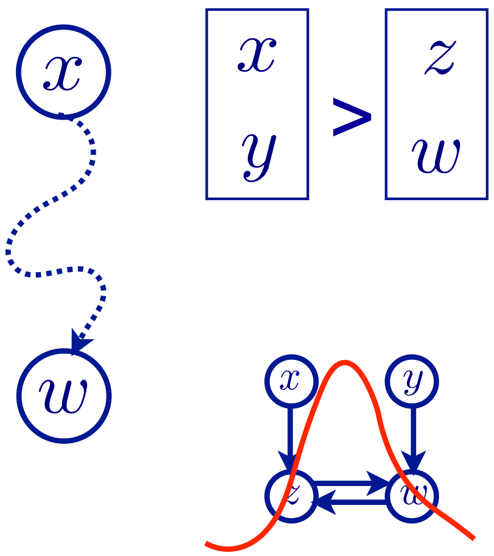


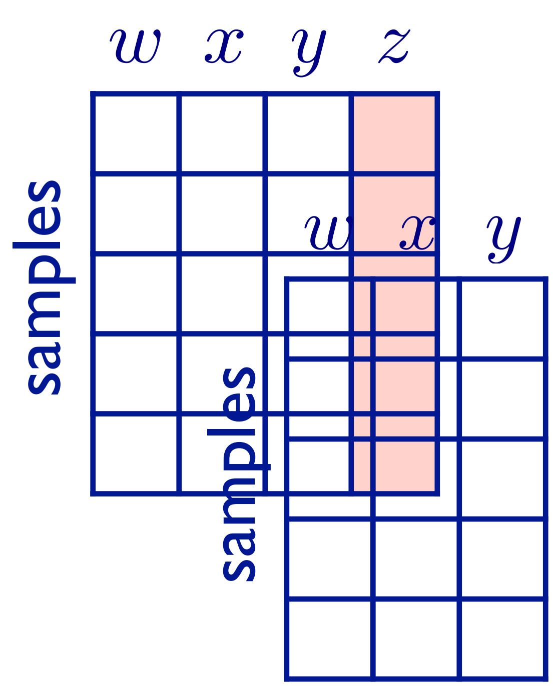
inference algorithm

Causal Discovery

background knowledge



- edge presences/ absences
- pathways
- tier orderings
- "priors"



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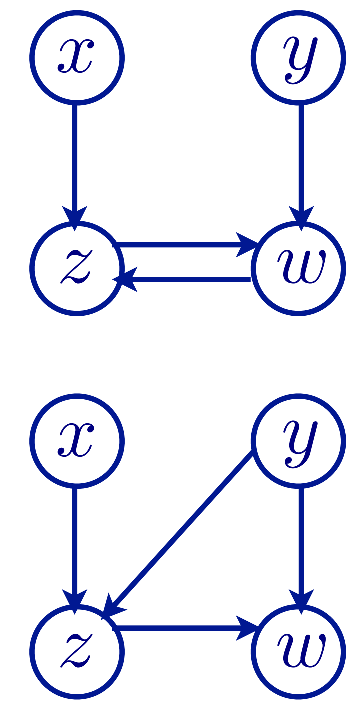
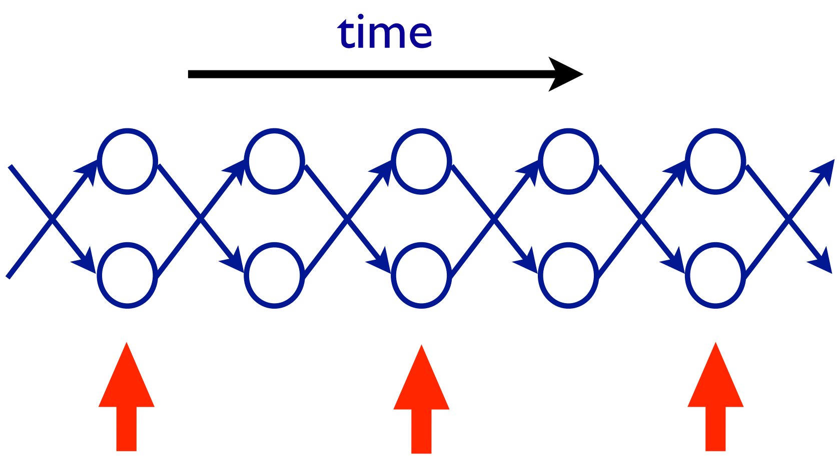


etc.

Settings: examples

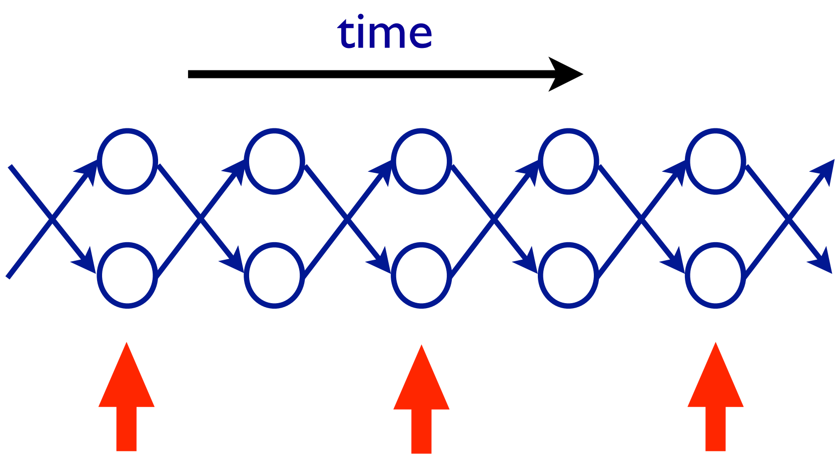
subsampled time series

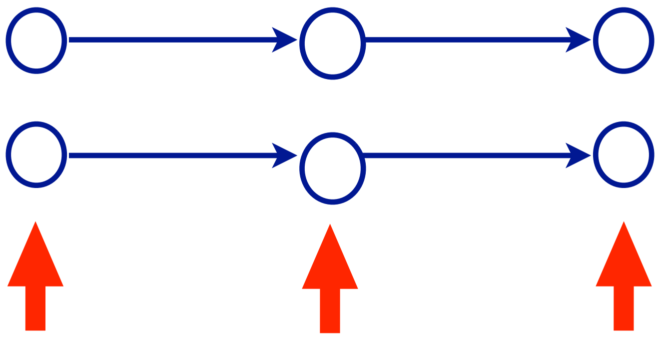


(cf. work by Plis & Danks)

Settings: examples

subsampled time series





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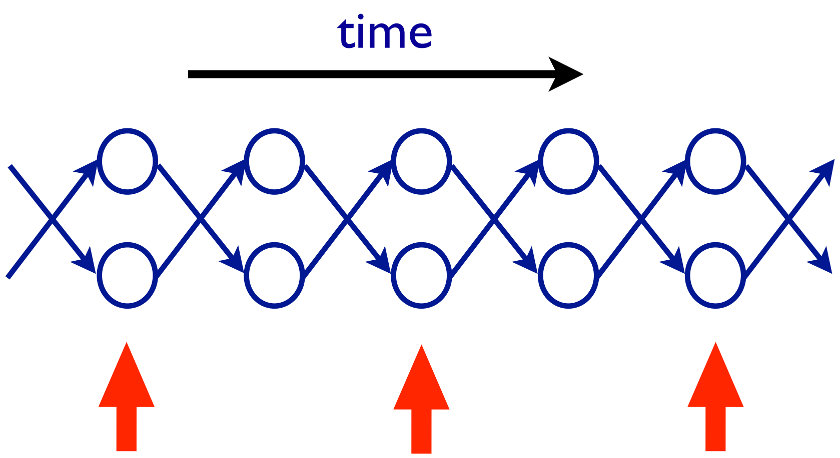

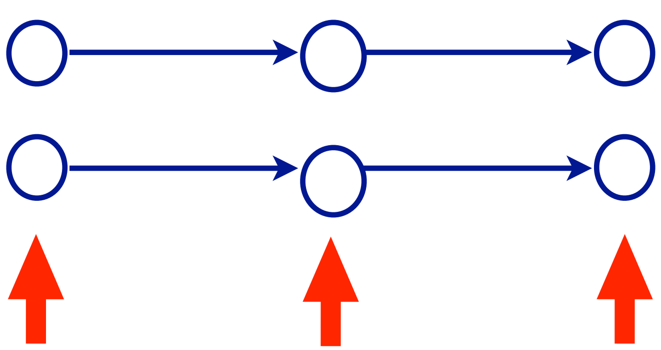

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inference algorithm

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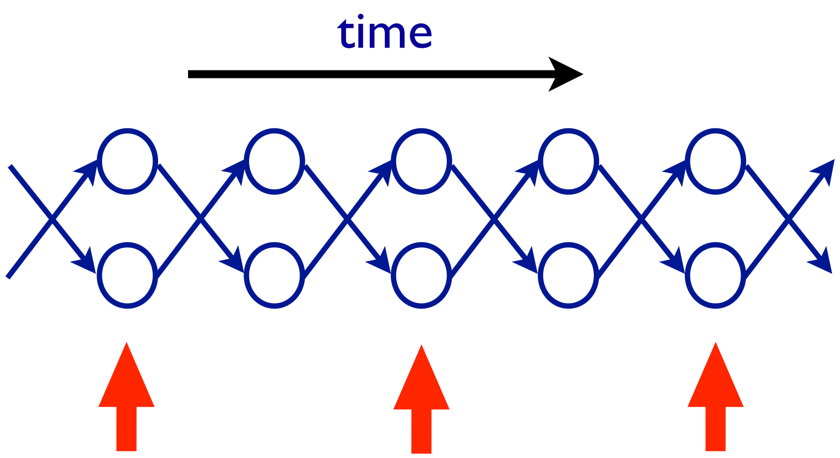

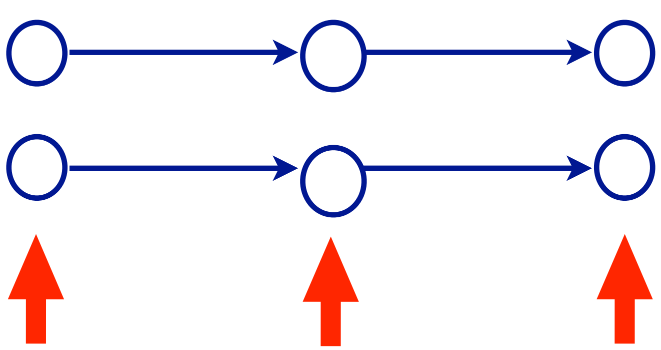

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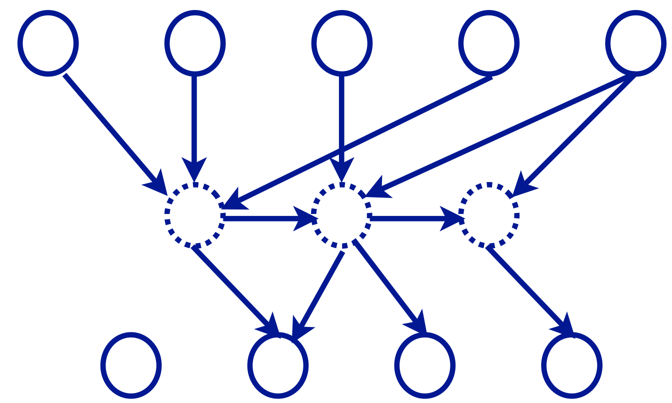
biological settings



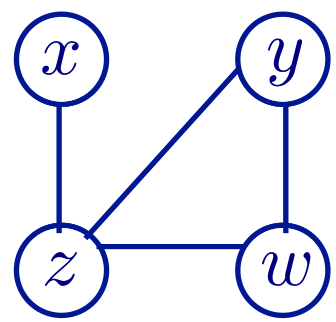
(cf. work by Murray-Watters & Glymour)

inference algorithm

Causal Effects

inference algorithm

Causal Effects

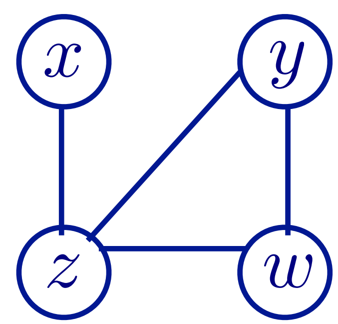


inference algorithm

equivalence class

Causal Effects

inference algorithm



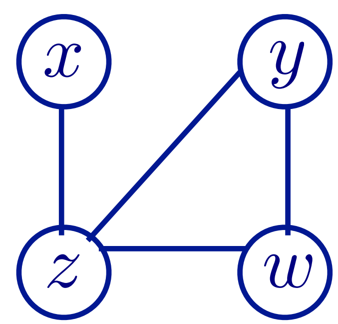
equivalence class

$$P(y|do(w))$$

causal effect

Causal Effects

inference algorithm



equivalence class

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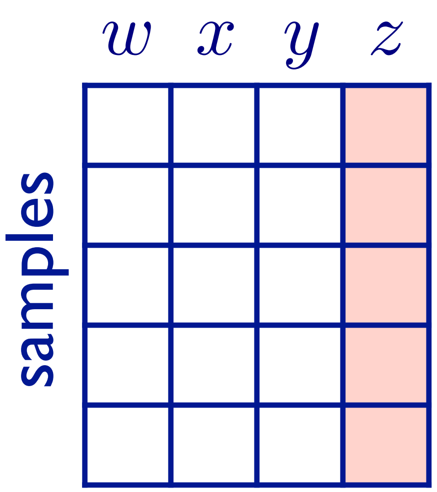
causal effect

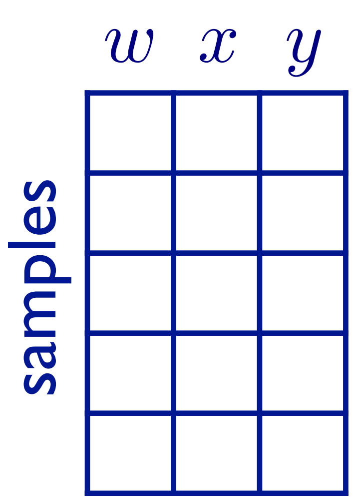
How to apply the do-calculus in settings when the causal structure is underdetermined?

High-Level

High-Level

data
sample





High-Level

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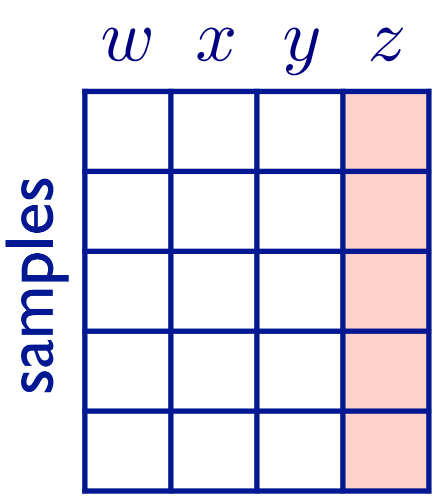
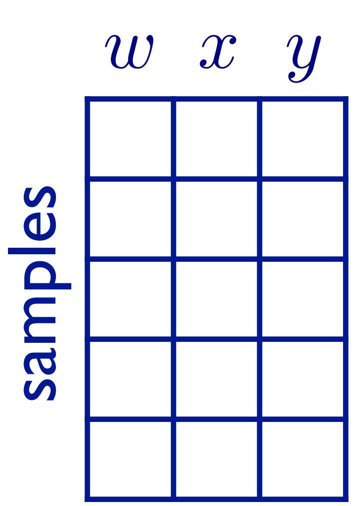
- etc.





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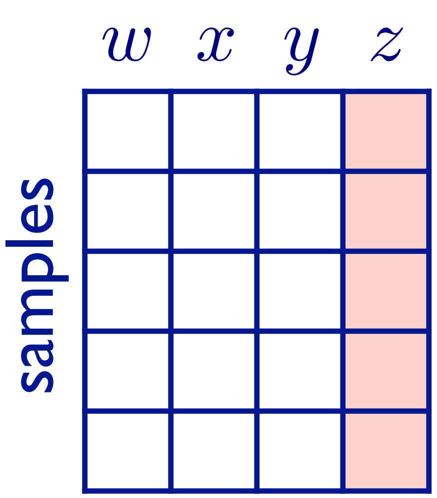

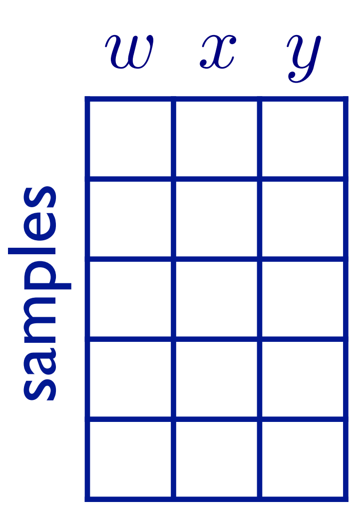

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background knowledge, e.g.
- pathways
- tier ordering
- “priors”
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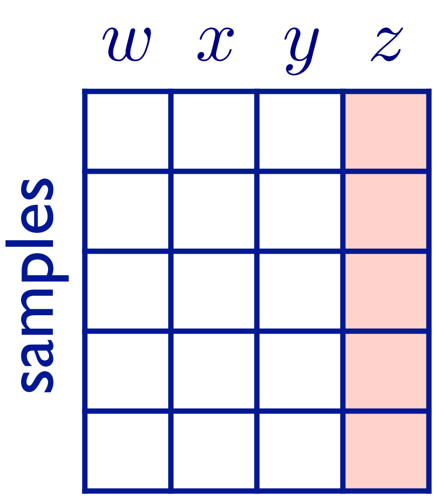
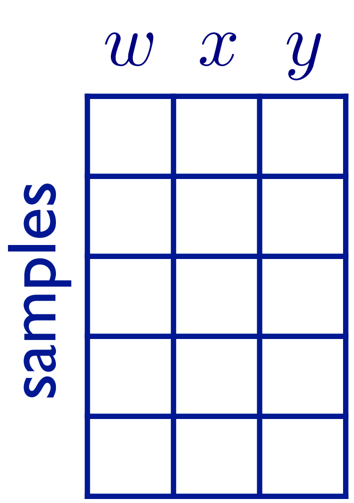
setting
- subsampled time series
- tier structure





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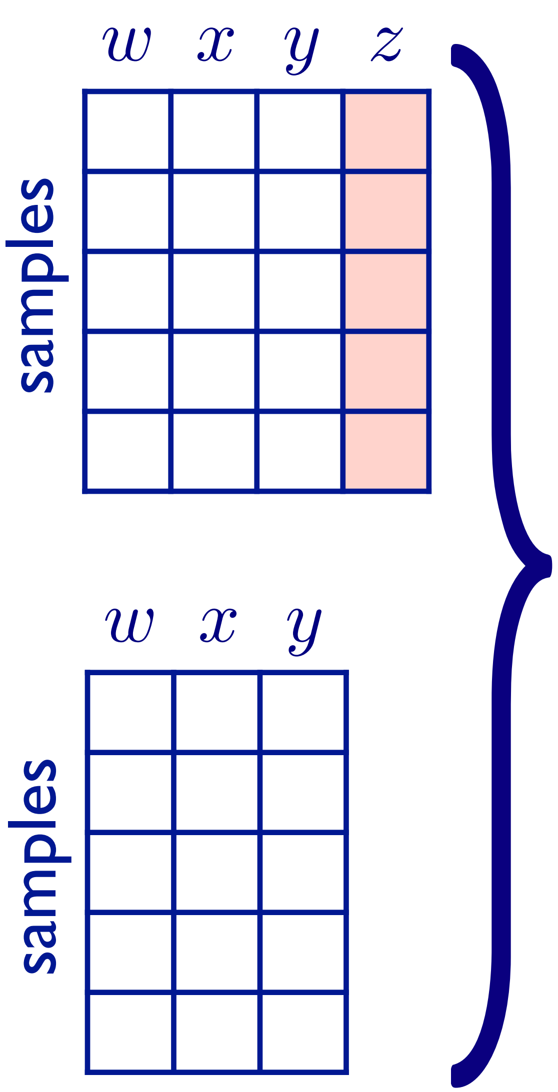
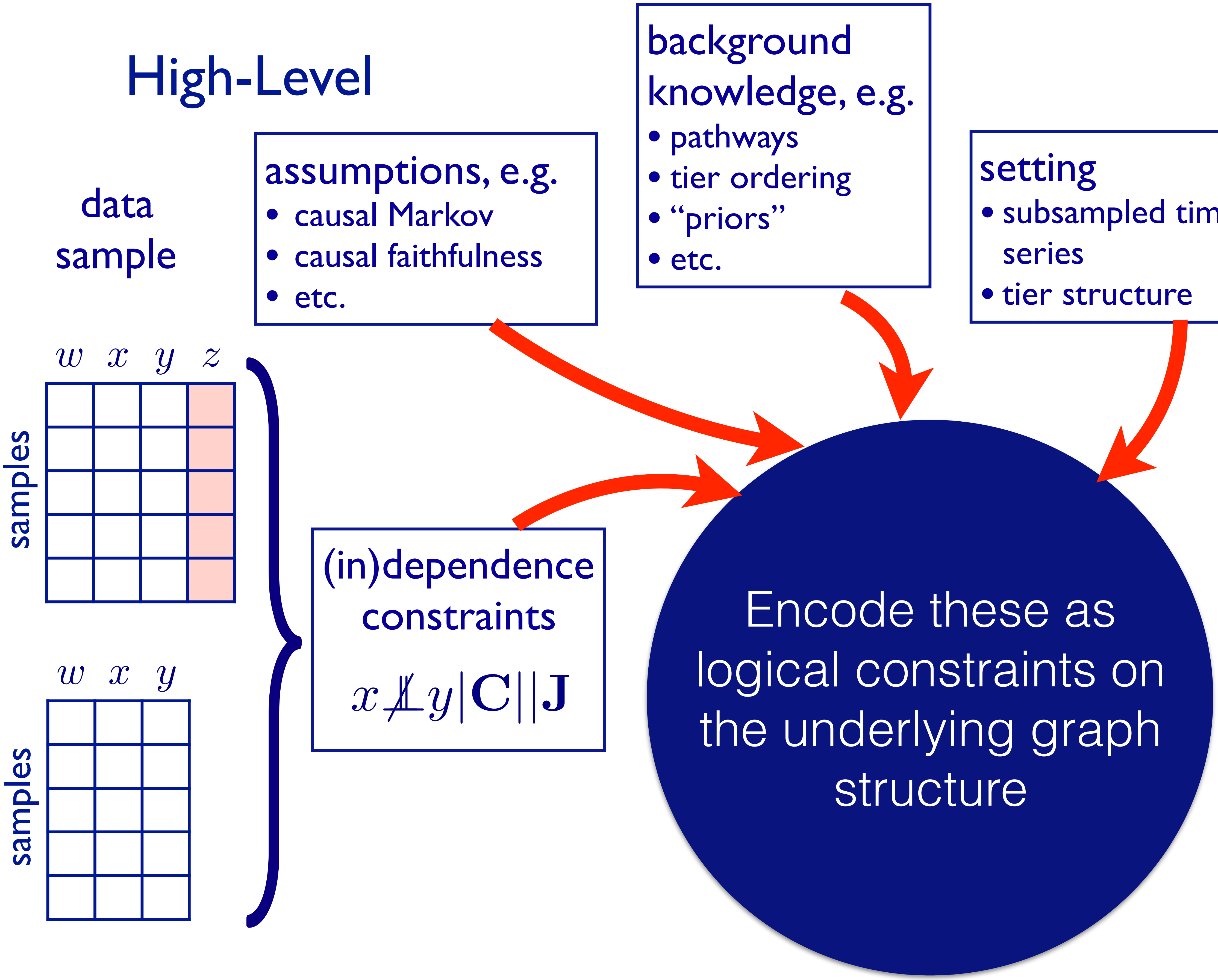


(in)dependence constraints

$$x \not\perp\!\!\!\perp y \mid \mathbf{C} \,||\, \mathbf{J}$$

High-Level


data sample
w x y z
samples
w x y
samples
assumptions, e.g.
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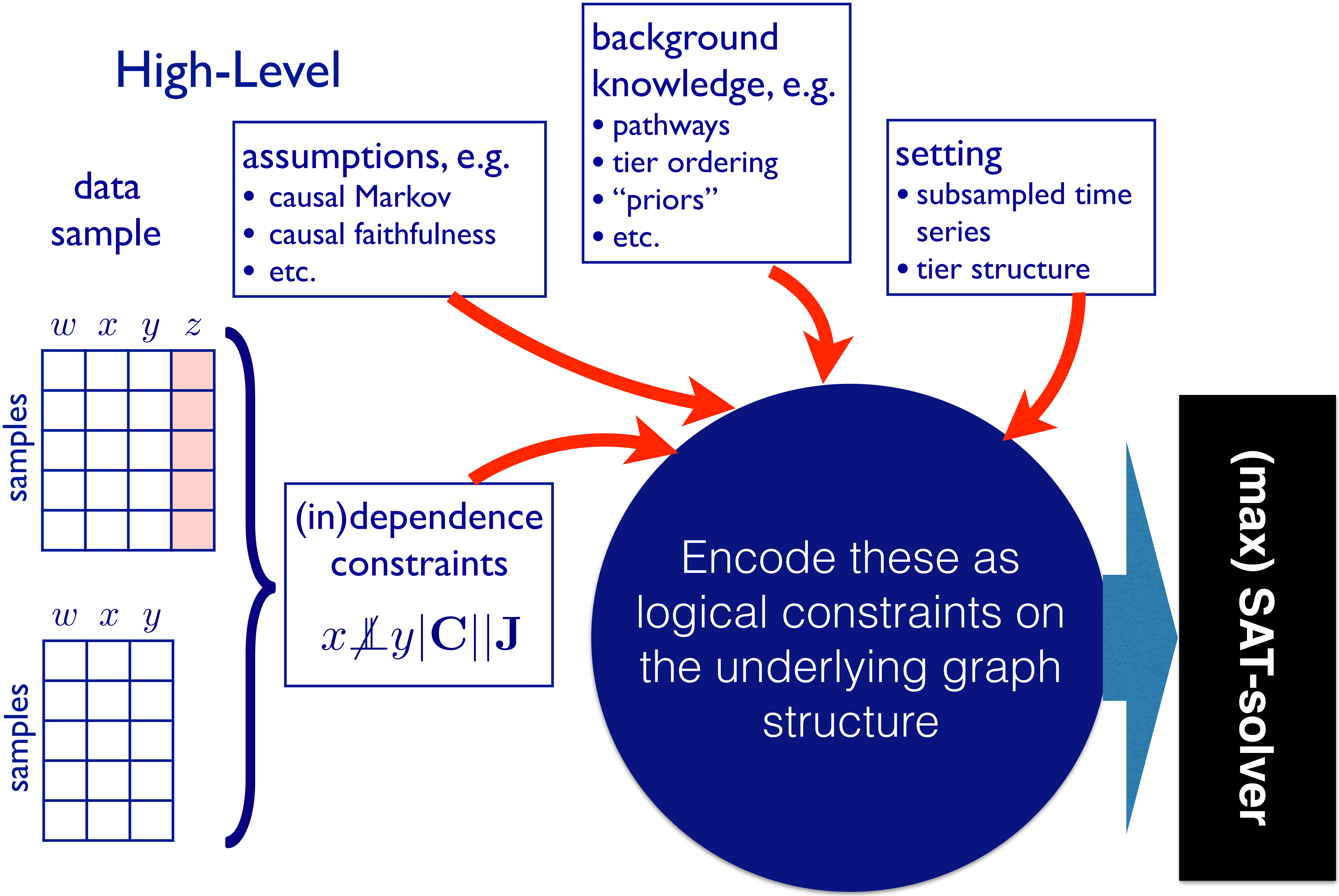
(in)dependence constraints
$x \not\perp\!\!\!\perp y \,|\, \mathbf{C} \,||\, \mathbf{J}$
Encode these as logical constraints on the underlying graph structure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$x \not\perp\!\!\!\perp y \,|\, \mathbf{C} \,||\, \mathbf{J}$
Encode these as logical constraints on the underlying graph structure
(max) SAT-solver

d-separation and independence

- Under the assumption of causal Markov and causal Faithfulness:

$$x \not\perp y \mid \mathbf{C} \;||\; \mathbf{J} \iff x \not\perp\!\!\!\perp y \mid \mathbf{C} \;||\; \mathbf{J}$$

d-separation and independence

- Under the assumption of causal Markov and causal Faithfulness:

$$x \not\perp y \mid \mathbf{C} \,||\, \mathbf{J} \iff x \not\perp\!\!\!\perp y \mid \mathbf{C} \,||\, \mathbf{J}$$

x and y are d-connected given **C** when variables in **J** are subject to intervention $\iff$ x and y are dependent given **C** when variables in **J** are subject to intervention

Example with 3 variables: x, y, z

$$x \perp\!\!\!\perp y$$

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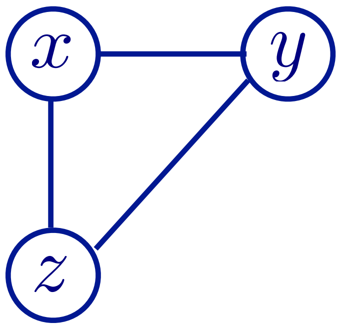
PC-algorithm



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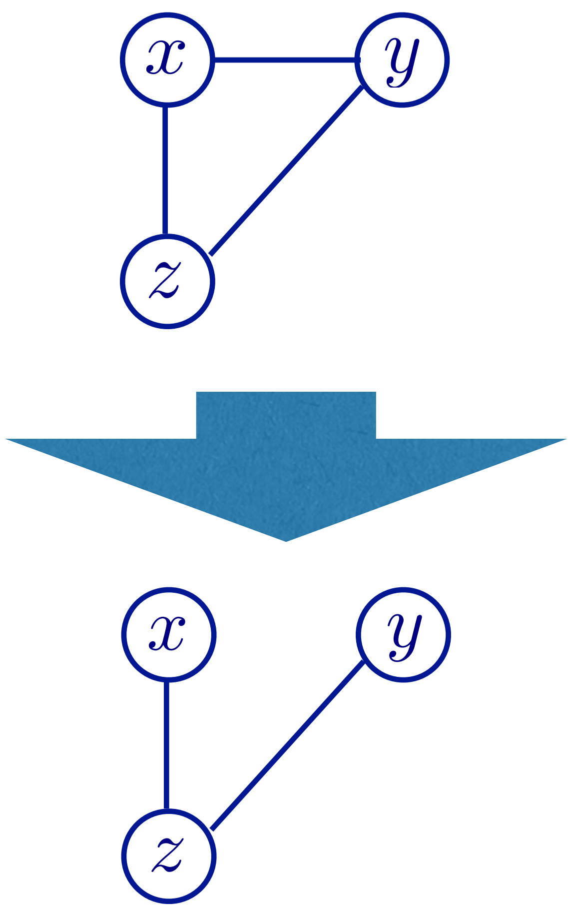

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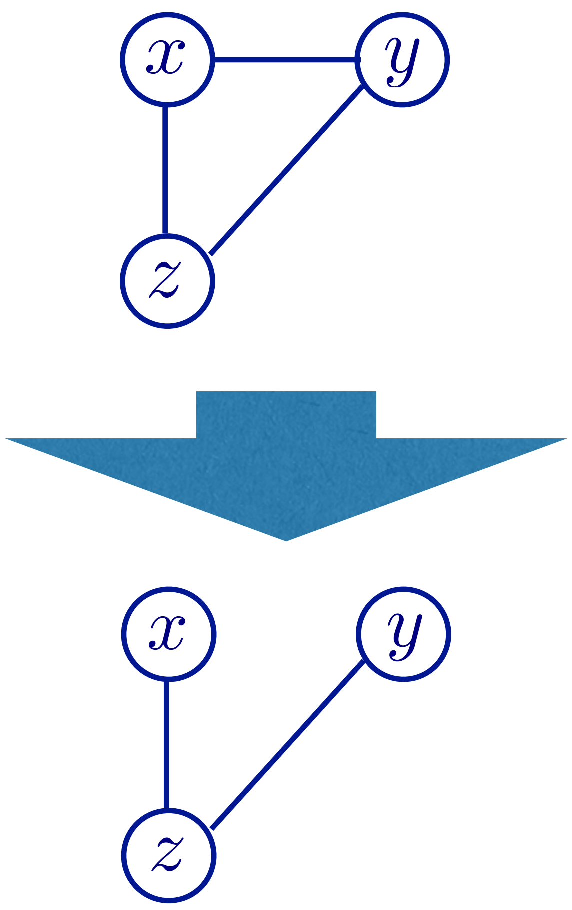


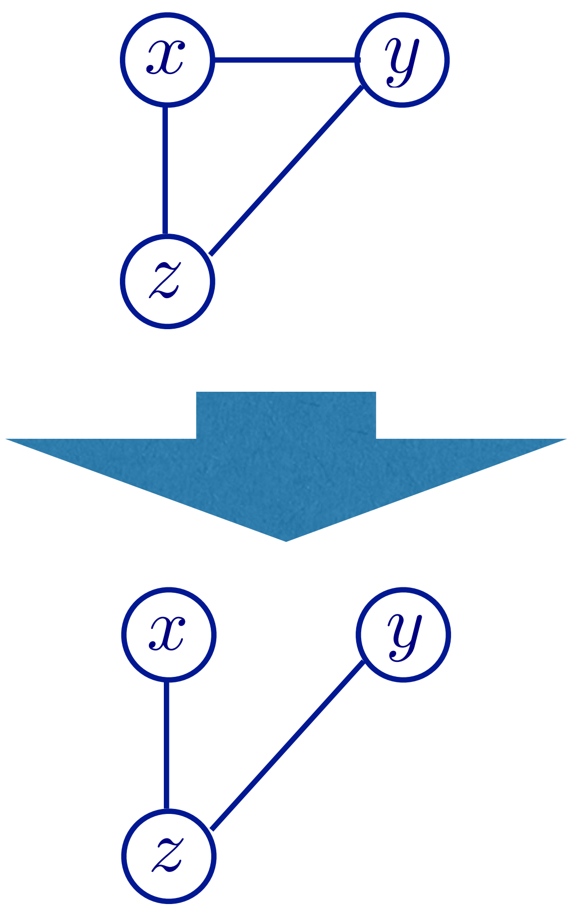
SAT-algorithm

define atoms
$$\begin{cases} A := \text{``}x \to y \in G\text{''} \\ B := \text{``}y \to x \in G\text{''} \\ C := \text{``}z \to x \in G\text{''} \\ D := \text{``}z \to y \in G\text{''} \\ \quad \ldots \end{cases}$$

Example with 3 variables: x, y, z

$$x \perp\!\!\!\perp y$$

PC-algorithm



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encode constraint

$$\begin{cases} \neg A \wedge \neg B & \text{// direct edges} \\ \wedge \neg (C \wedge D) & \text{// common causes} \\ \wedge \neg \ldots & \text{// indirect paths} \end{cases}$$

SAT-based Causal Discovery

- **Formulate the independence constraints in propositional logic**

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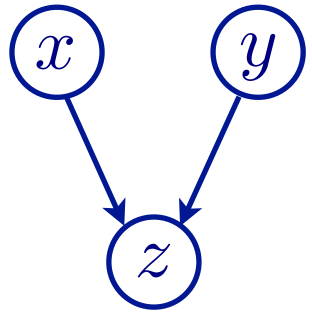

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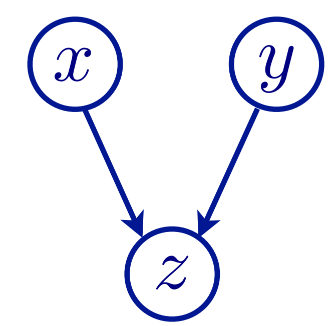

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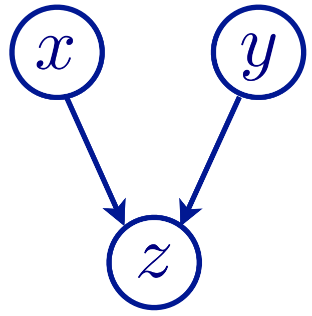

➡ very general setting (allows for cycles and latents) and trivially complete

➡ **BUT**: erroneous test results induce conflicting constraints: UNsatisfiable

Conflicts and Errors

- **Statistical independence tests produce errors**

constraint

$$x \not\perp\!\!\!\perp z$$

$$y \not\perp\!\!\!\perp z$$

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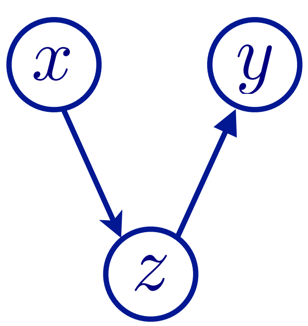

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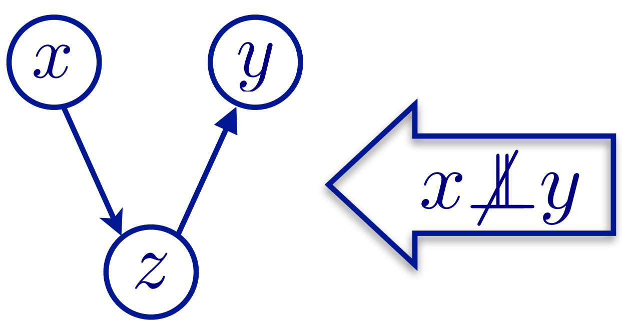

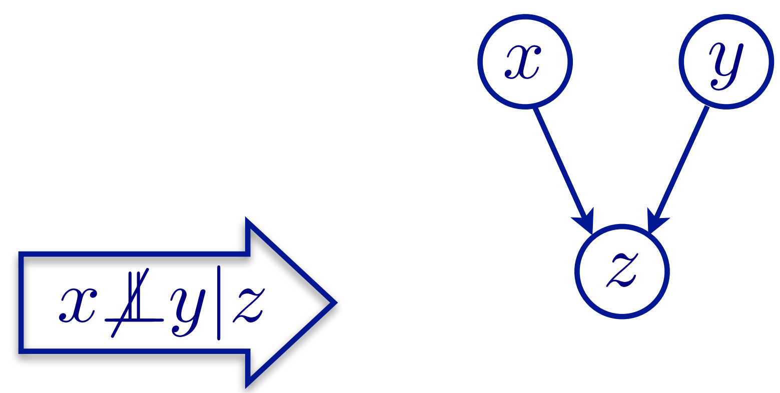

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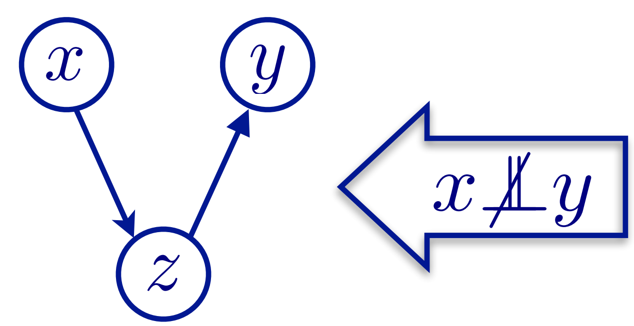
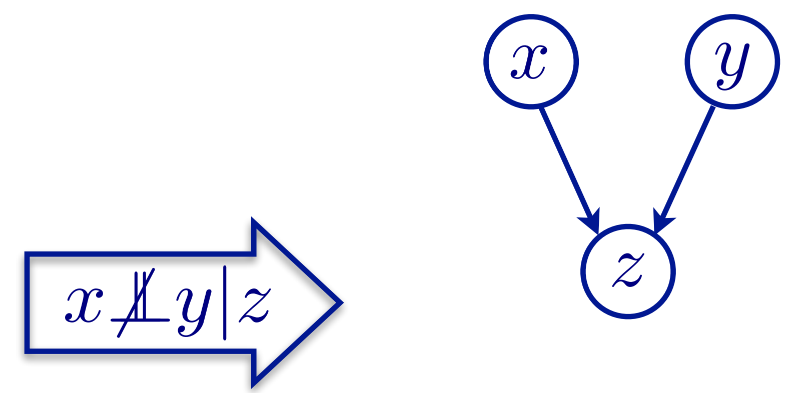


constraint	weight
$x \not\perp\!\!\!\perp z$	3000
$y \not\perp\!\!\!\perp z$	2500
$x \perp\!\!\!\perp y$	500
$x \perp\!\!\!\perp y \mid z$	250



Constraint Satisfaction Approach

- **INPUT: (in)dependence constraints weighted according to reliability**

$$\min_{G} \sum_{k\,:\,\text{constraint } k \text{ is } \textbf{not} \text{ satisfied by } G} w(k)$$

- **OUTPUT: a graph G that minimizes the cost**

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What are suitable weights?

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$$\begin{array}{ccc} x \not\perp\!\!\!\perp y \mid C & & x \perp\!\!\!\perp y \mid C \\ P(x|C)P(y|x,C) & \text{vs.} & P(x|C)P(y|C) \end{array}$$

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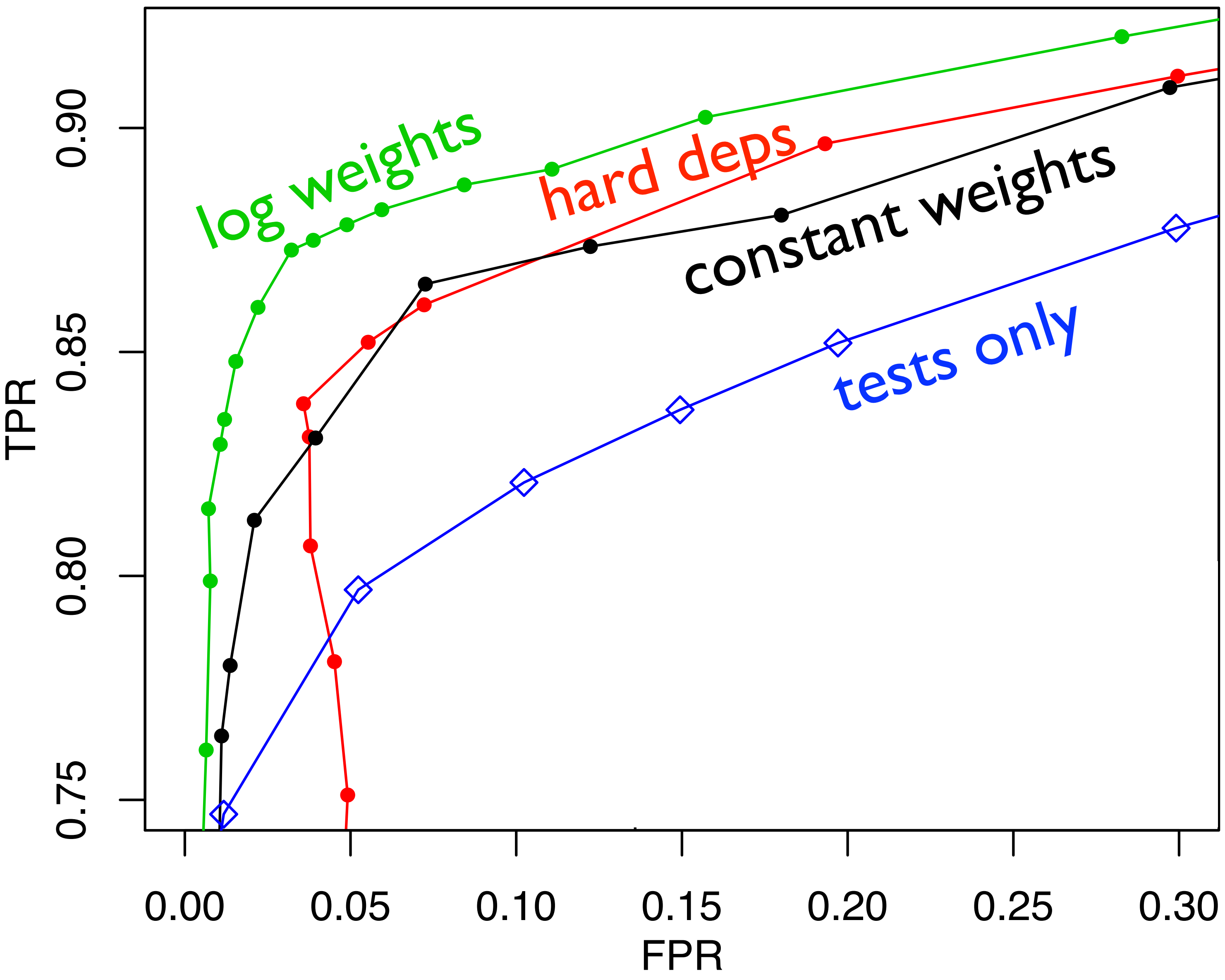
  - probabilistic classifier: find G such that if it were true, test results would be optimal in the sense of a proper score

Optimization

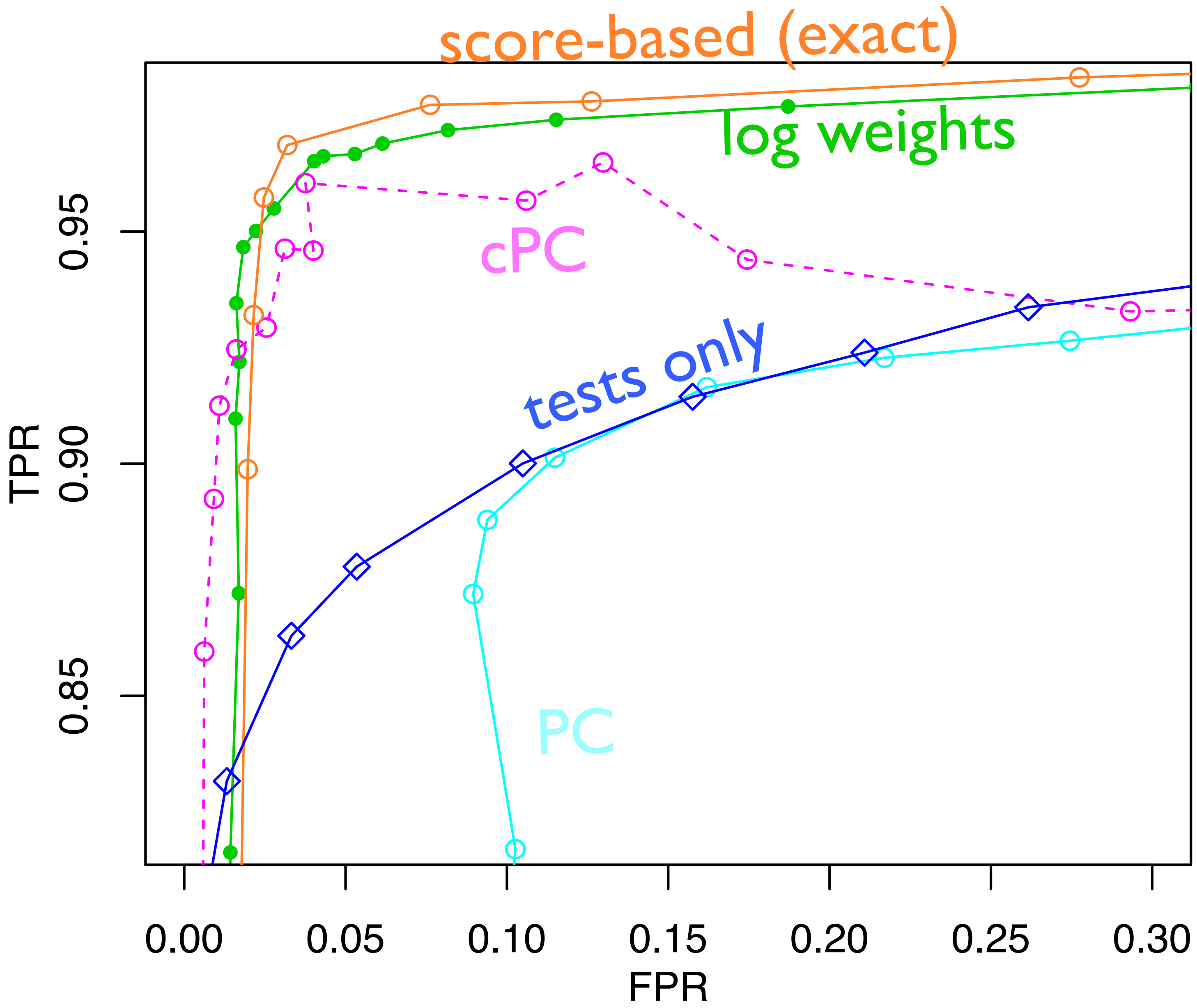
- Answer Set Programming (ASP) is a modern declarative programming paradigm
  - solver used: Clingo
  - SAT-solver and branch and bound algorithm
  - finds globally optimal weighted maxSAT solution

Simulation 1: cycles and latents



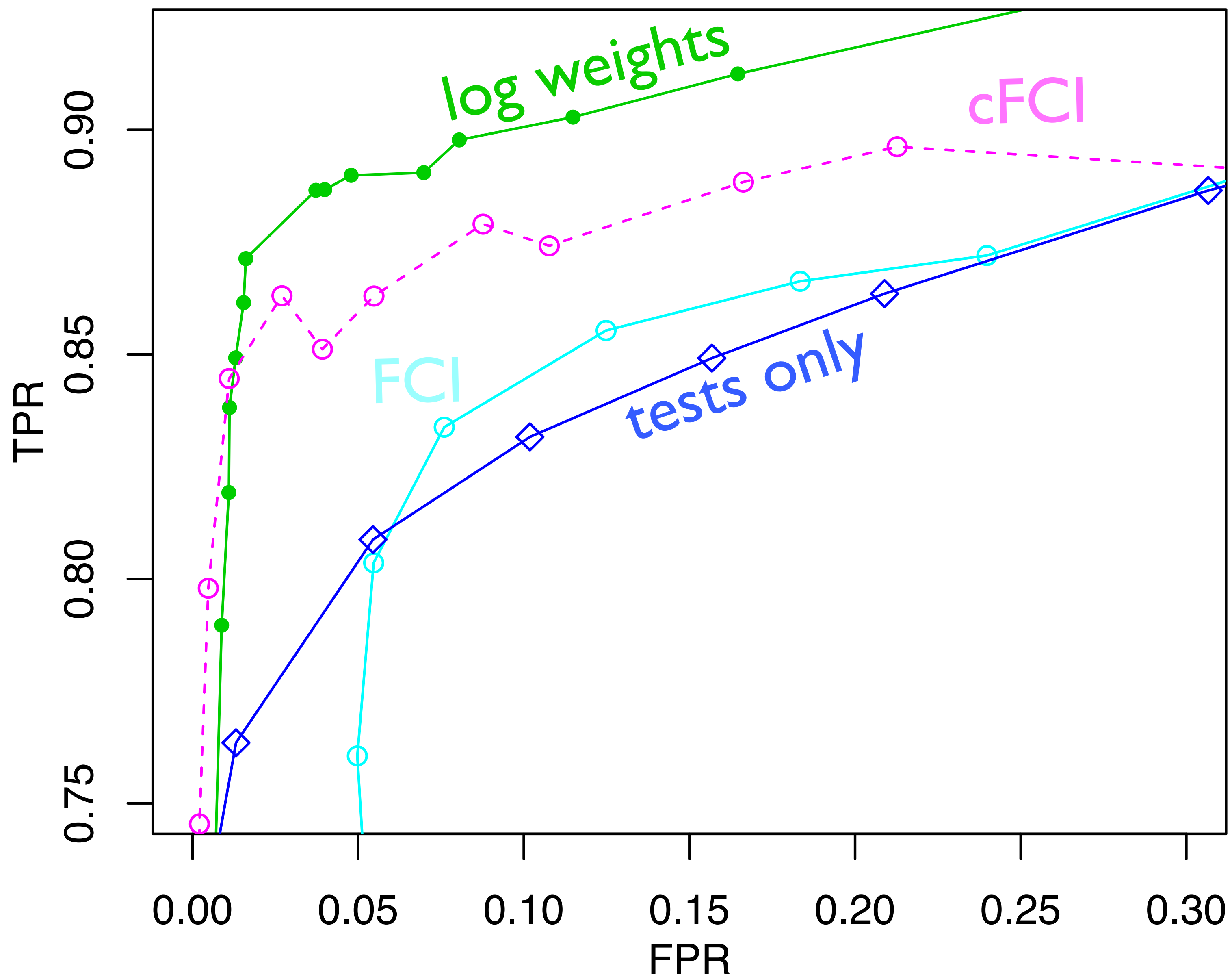
- ROC of dependences, passive observational data set, 6 observed variables, average degree 2; 500 samples, 200 models, linear Gaussian parameterization

Simulation 2: no cycles, no latents



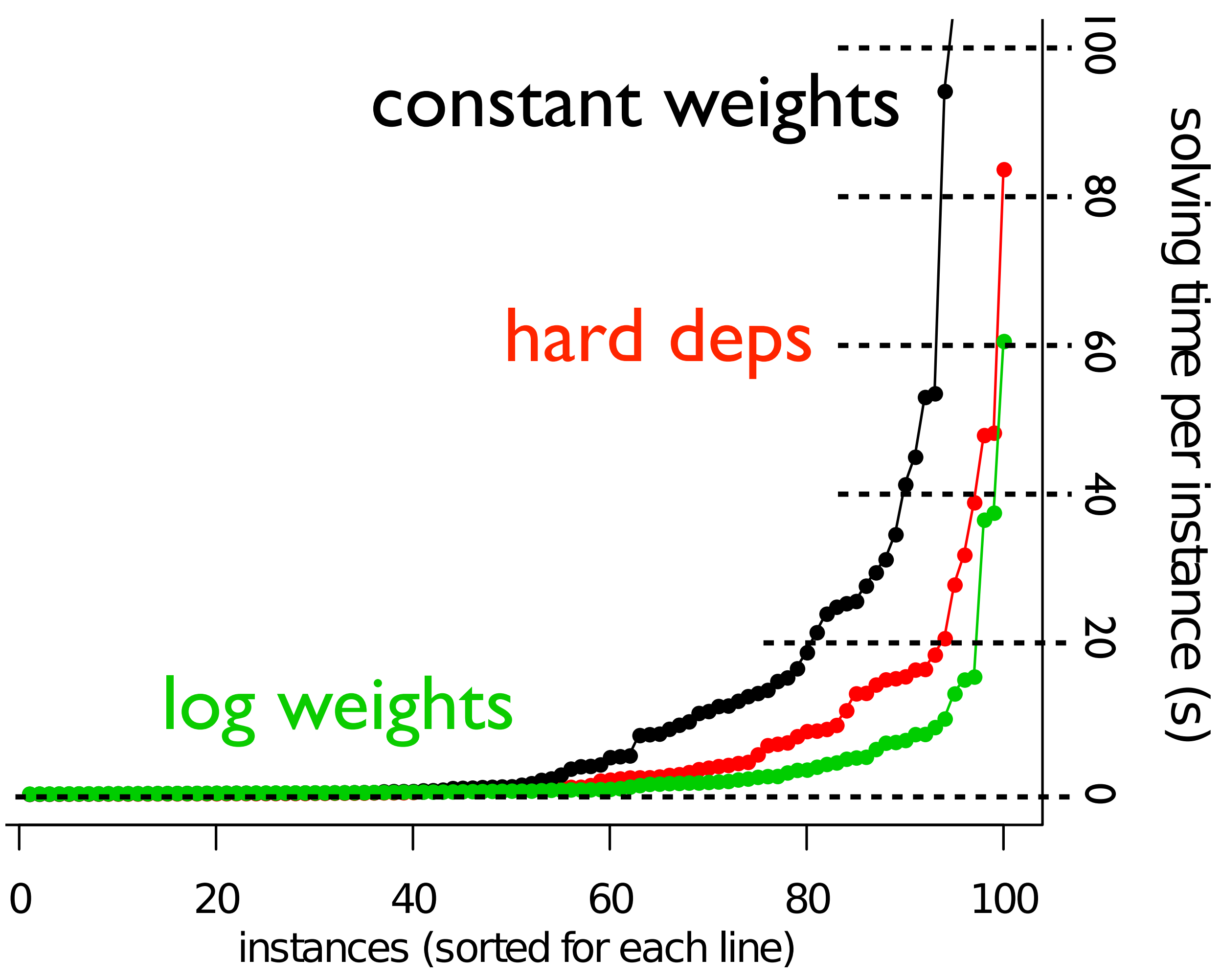
- cPC returns a fully determined output only 58/200 times at its optimum

Simulation 3: no cycles, but latents



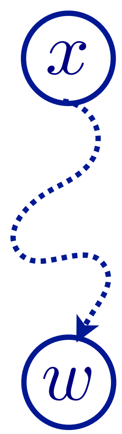
- cFCI only returns unambiguous results 61/200 times at its optimum

Simulation 4: Scalability



- up to 7 variables and only a few data sets for now (9x10^18 graphs)

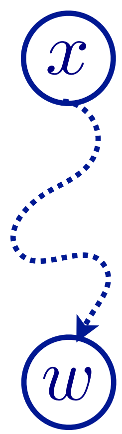
Background Knowledge



$$\boxed{\begin{matrix} x \\ y \end{matrix}} > \boxed{\begin{matrix} z \\ w \end{matrix}}$$

“priors”

Background Knowledge



$$\Longleftrightarrow \qquad x \not\perp\!\!\!\perp w \,||\, x$$

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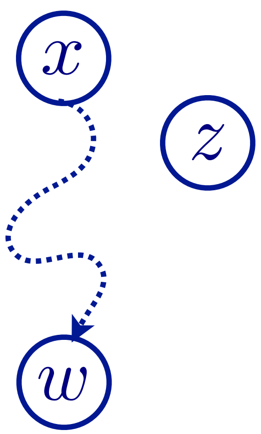

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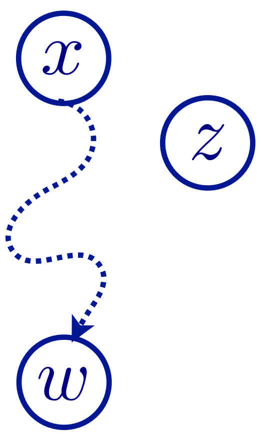

$$\Longleftrightarrow \qquad x \not\perp\!\!\!\perp w \,||\, x \; z \qquad weight = 0.8$$

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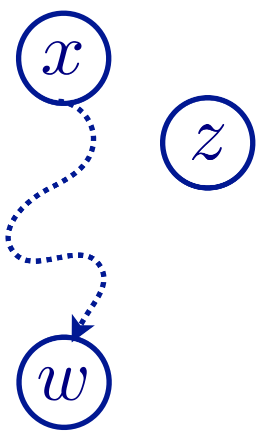

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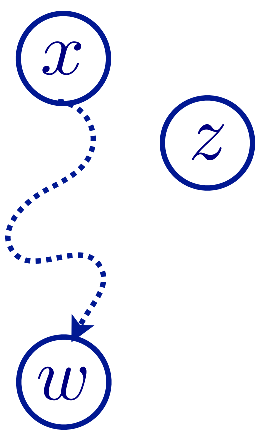

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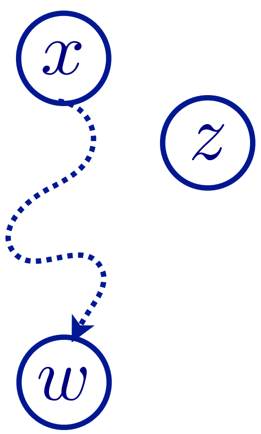

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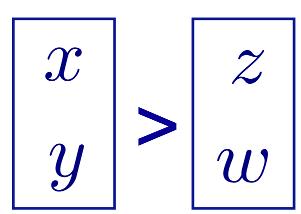

$\Longleftrightarrow$

$$(x > z) \wedge (x > w)$$
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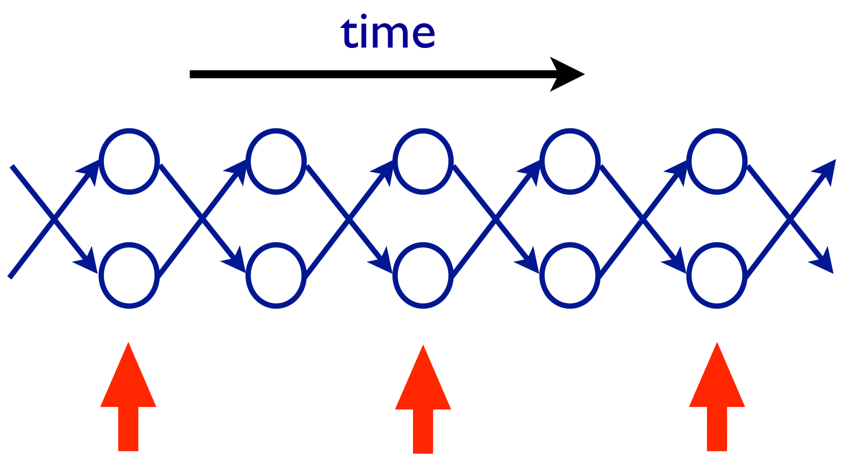
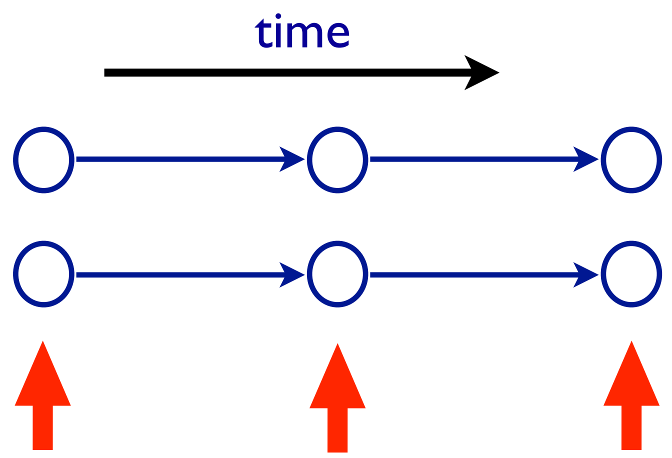
“priors”

$\Longleftrightarrow$

- specific probabilities for each graph
- soft sparsity constraint
- …

Settings





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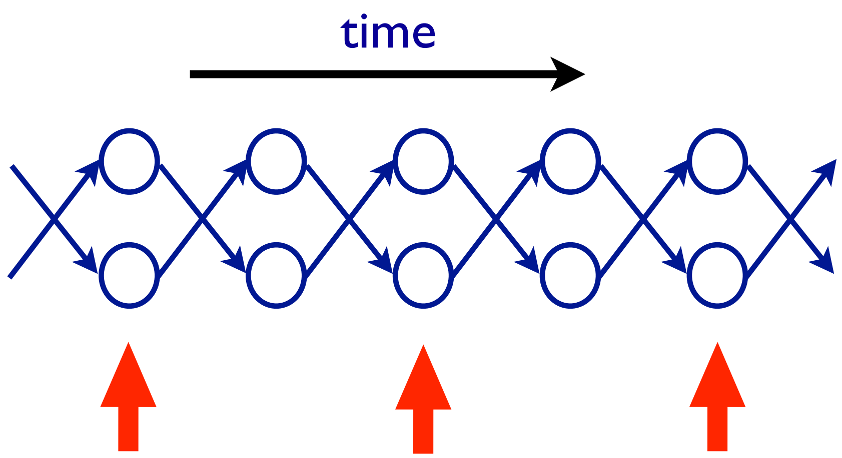

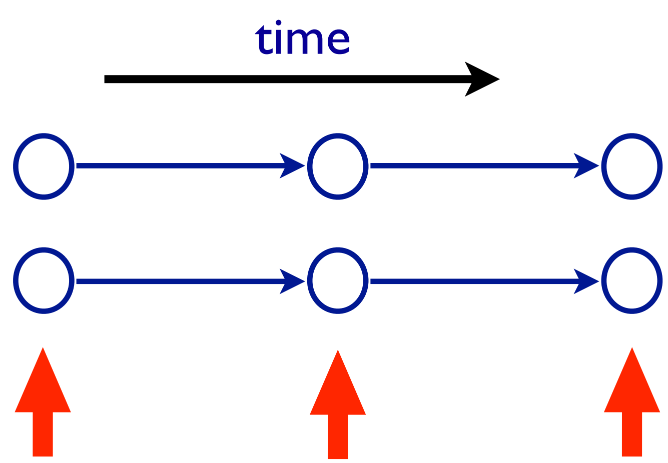


```
urange(1..5).

1 { u(U): urange(U) } 1.

{ edge1(X,Y) } :- node(X), node(Y).

path(X,Y,1) :- edge1(X,Y).
path(X,Y,L) :- path(X,Z,L-1), edge1(Z,Y),
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X < Y, L < U, u(U).

:- edgeu(X,Y), not edge1u(X,Y).
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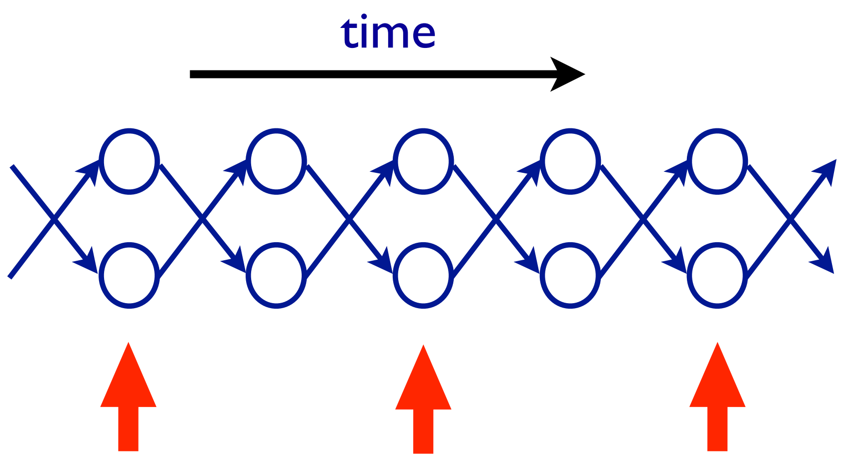

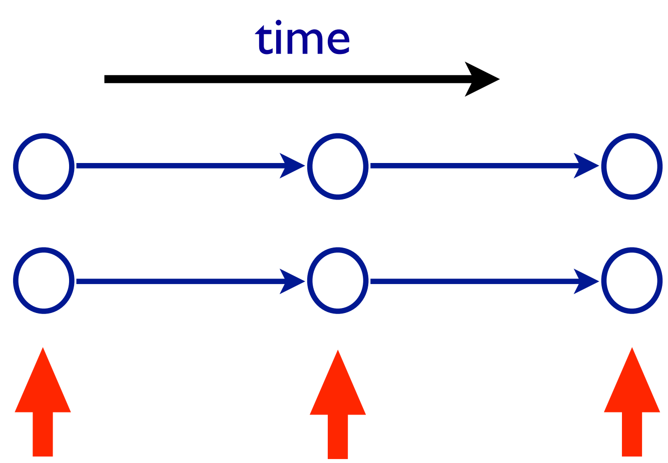

range for rate of subsampling

```
urange(1..5).
```

subsampling rate is unique

```
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def. of edge in graph

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recursive def. of path

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def. of edge in subsampled graph

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def. of how confounders arise due to subsampling

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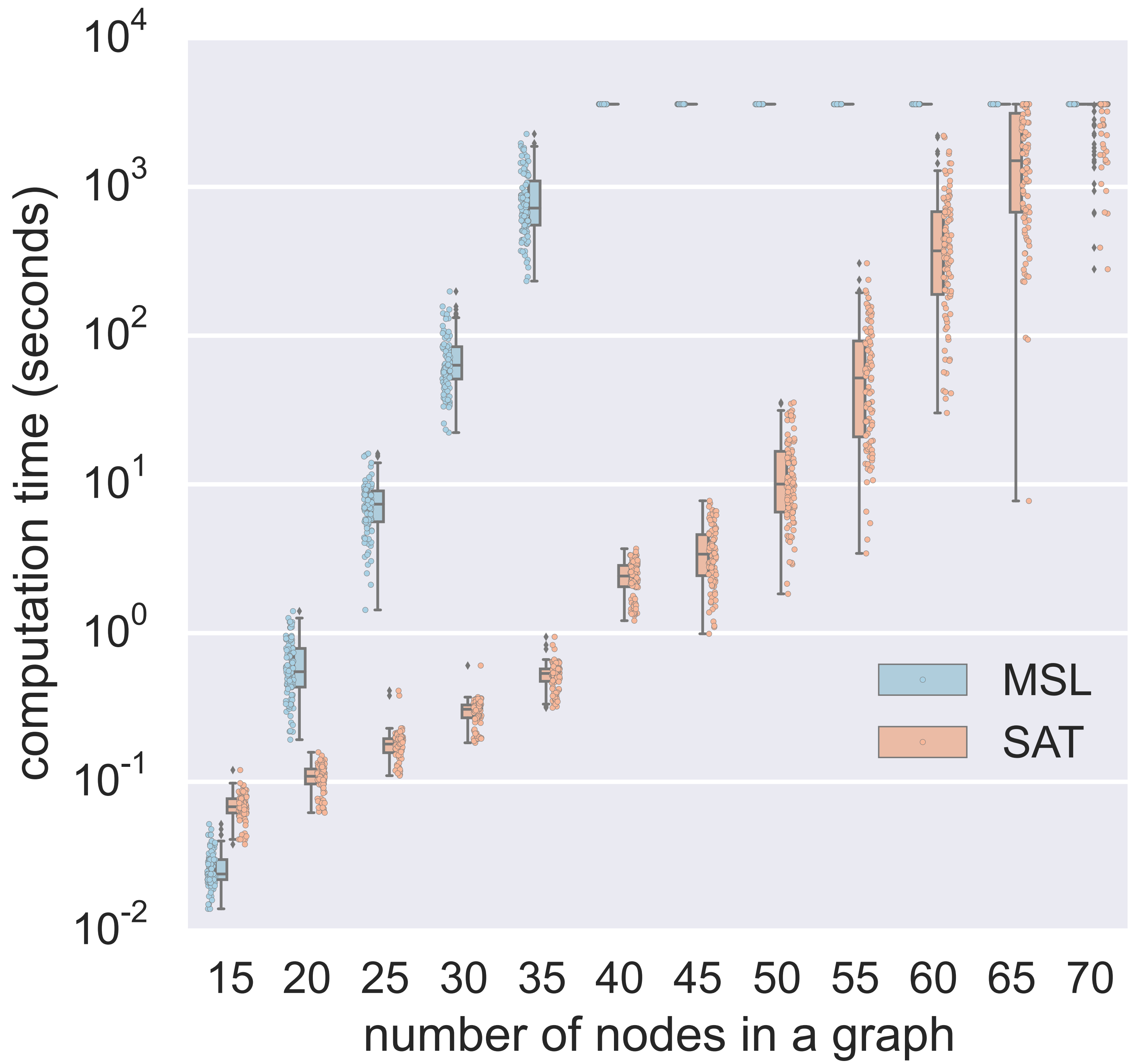
```

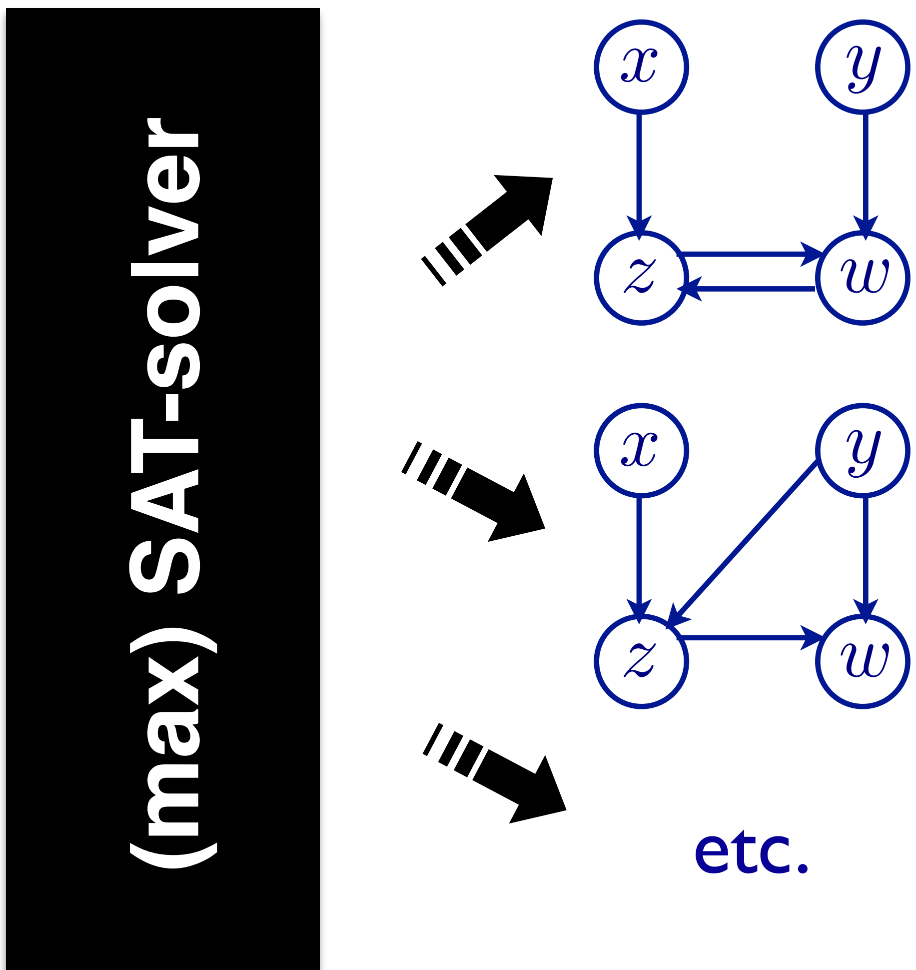
constraints on how edges in subsampled graph relate to edges in true graph

Runtime comparison

For a graph determined at subsampling rate 2, infer the equivalence class of graphs at the system time scale (1-step)



Output of Causal Search Algorithms



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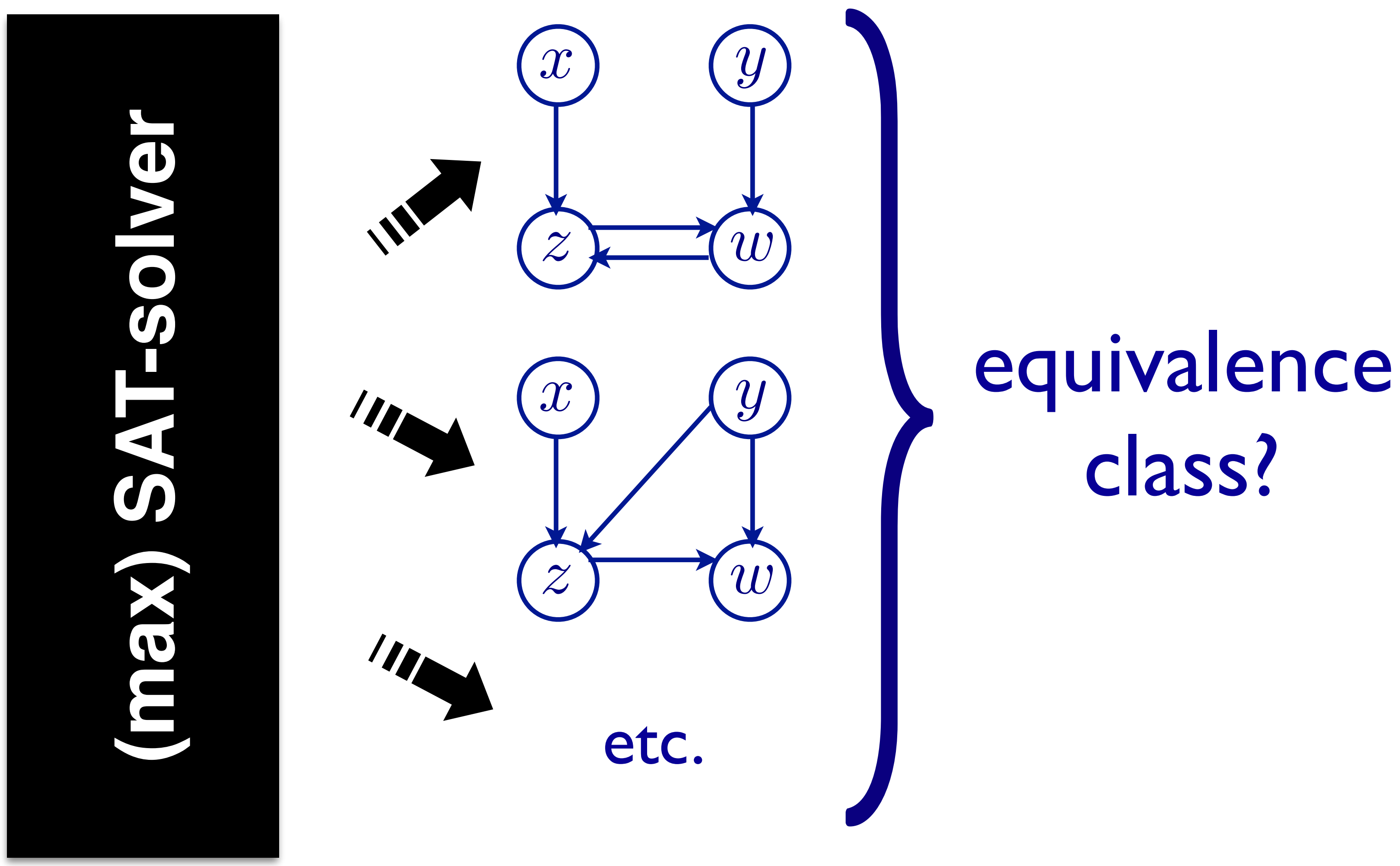

Output of Causal Search Algorithms

Query:

(max) SAT-solver

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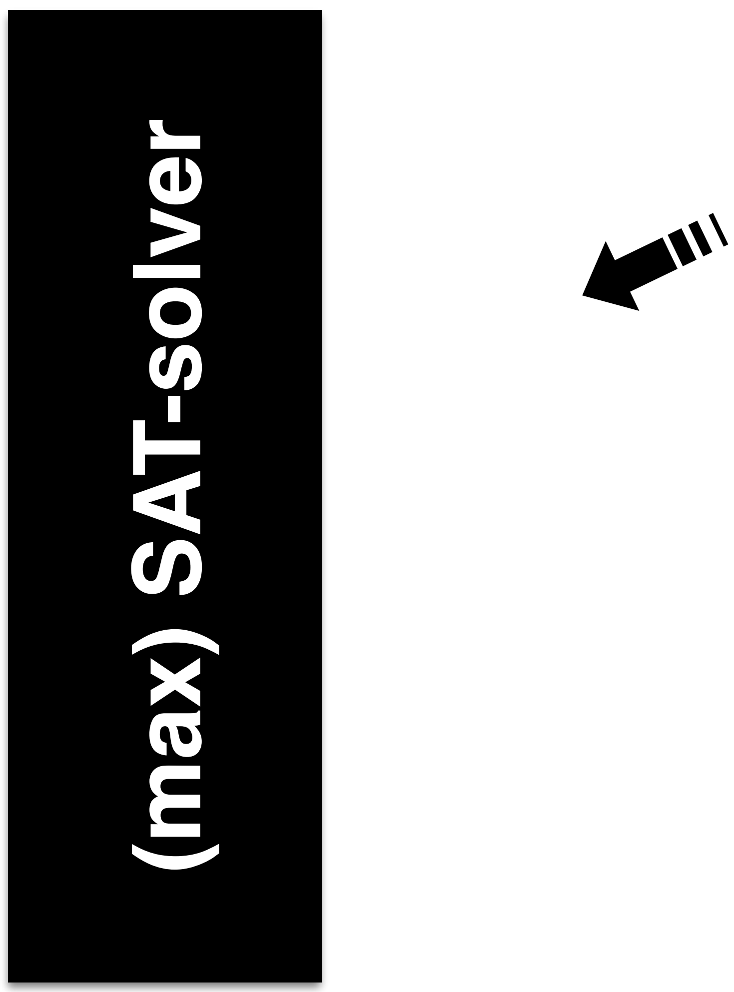

Query:

- list the structures in the equivalence class

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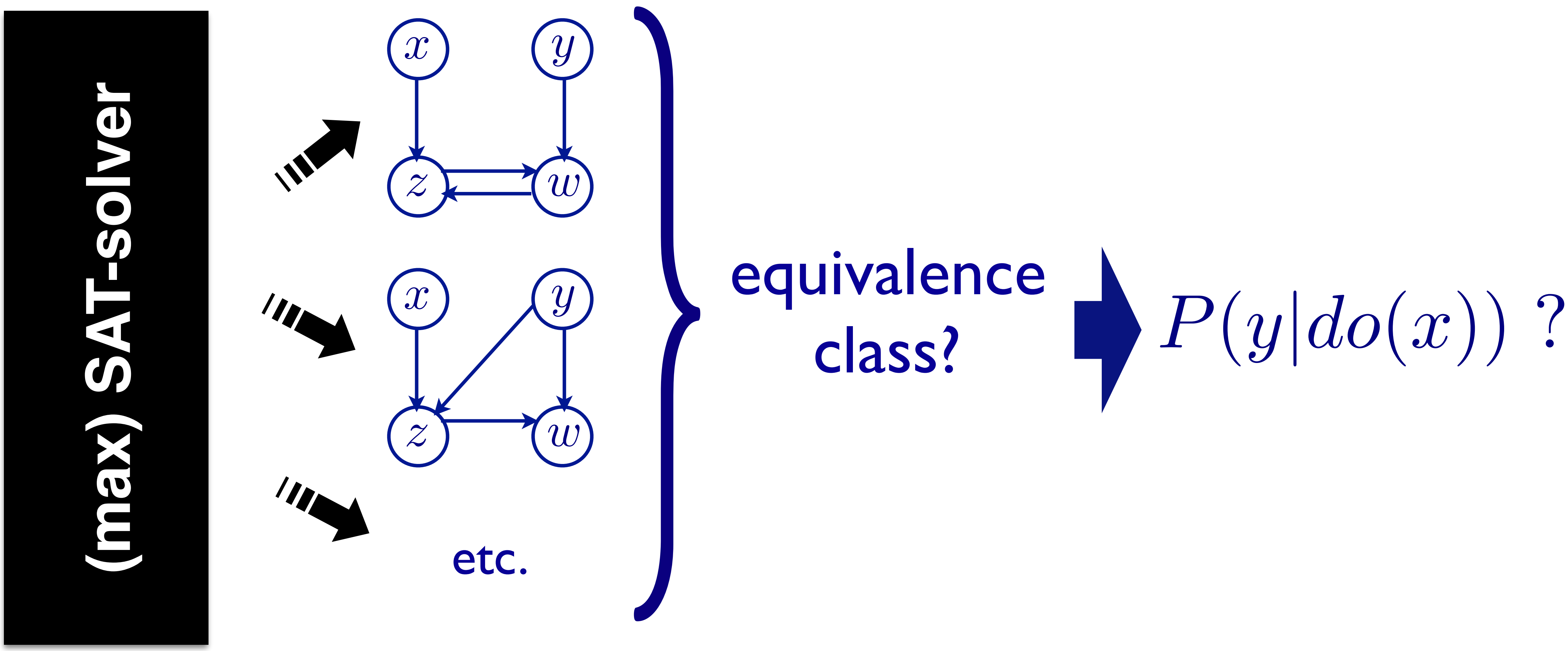
Response:

- enumeration of solutions
- “backbone” of the SAT-instance
- …

Computing Causal Effects



equivalence class? ➡ $P(y|do(x))$?

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- enumerate each graph in the equivalence class and run the Tian-Shpitser algorithm to determine the causal effect?

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- Alternative:

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do-calculus

Rule 1 (insertion/deletion of observations)

$$P(y|do(x), z, w) = P(y|do(x), w) \text{ if } Y \perp\!\!\!\perp Z|X, W||X$$

Rule 2 (action/observation exchange)

$$P(y|do(x), do(z), w) = P(y|do(x), z, w) \text{ if } Y \perp\!\!\!\perp I_Z|X, Z, W||X$$

Rule 3 (insertion/deletion of actions)

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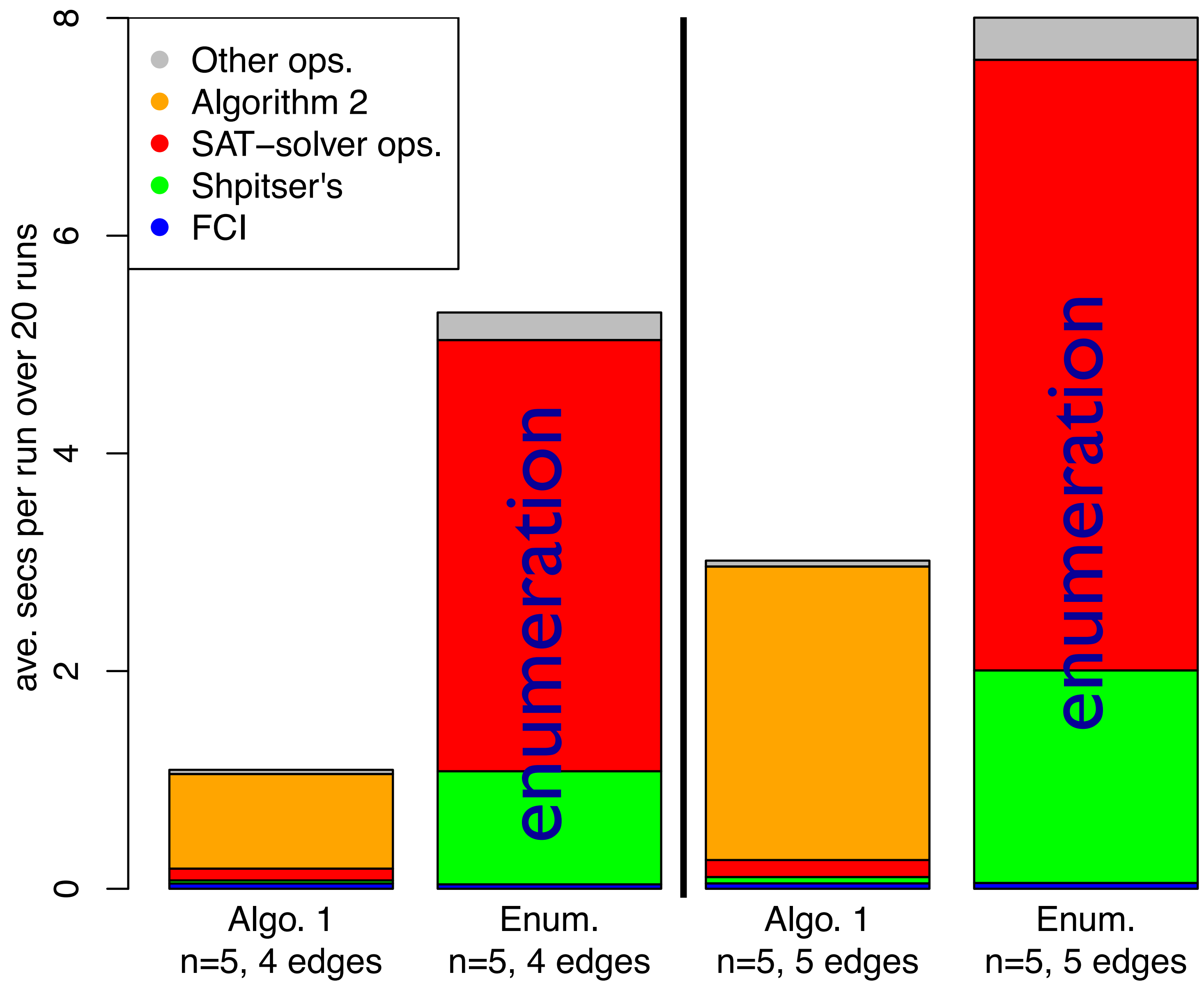
➡ search in the equivalence class over the possible applications of the *do*-calculus rules by *querying* the satisfaction of their conditions

Algorithm for the *do*-calculus when the graph is unknown

- determine the equivalence class implicitly using a SAT-solver
- query one solution graph G
- run the Tian-Shpitser-algorithm on G to determine whether the causal effect *P(y|do(w))* is determined for G
- if it is, determine which *do*-calculus rules were applied and record the constraints $C_1, \ldots, C_n$ that were used
  - add $\neg C_1 \vee \ldots \vee \neg C_n$ as a constraint to refine the current equivalence class
- if not, determine the "hedge" H and add $\neg H$ to refine the current equivalence class
- repeat until the equivalence is exhausted
- return the set of estimates of the causal effect and NA if it cannot be determined in one member of the equivalence class

Comparison of our approach to enumeration



In sum: *do*-calculus using a SAT-solver

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In sum: *do*-calculus using a SAT-solver

- enables computation of the causal effect when the graph structure is underdetermined

➡ how should one estimate a causal effect when the equivalence class of causal structures was determined on the basis of a set of conflicted constraints?

- some avenues one can explore with the query-based approach:
  - explore more closely the conditions involved in determining the causal effect
  - find multiple different estimators
  - even though the overall graph structure may not be determinable without resolving conflicts, some causal effects may be

Conclusion

- the use of general purpose SAT-solvers provides an extraordinarily versatile tool for causal discovery
- it opens new avenues for handling background knowledge and the computation of causal effects when the causal structure is underdetermined
- it provides a query based approach in contrast to a representation of an equivalence class of causal structures
- it suggests that current general purpose constraint solvers outperform domain specific approaches

References

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Other relevant work that is closely related:

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Thank you!