Nested Markov Models

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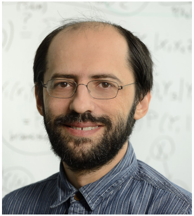


Ilya Shpitser
(Johns Hopkins)

Outline

- Part One: Non-parametric Identification
- Part Two: The Nested Markov Model

Part One: Non-parametric identification

- The general identification problem for DAGs with unobserved variables
- Simple examples
- Tian's Algorithm
- Formulation in terms of 'Fixing' operation

Intervention distributions (I)

Given a causal DAG $\mathcal{G}(V)$ with distribution:

$$p(V) = \prod_{v \in V} p(v \mid \text{pa}(v))$$

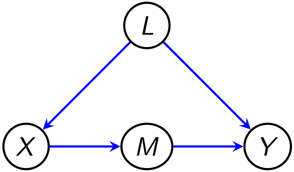
where $\text{pa}(v) = \{x \,|\, x \rightarrow v\}$;

Intervention distribution on X:

$$p(V \setminus X \mid \text{do}(X = \mathbf{x})) \quad = \quad \prod_{v \in V \setminus X} p(v \mid \text{pa}(v)).$$

here on the RHS a variable in X occurring in $\text{pa}(v)$, for some $v \in V \setminus X$, takes the corresponding value in $\mathbf{x}$.

Example



$$p(X, L, M, Y) = p(L)\ p(X \mid L)\ p(M \mid X) p(Y \mid L, M)$$

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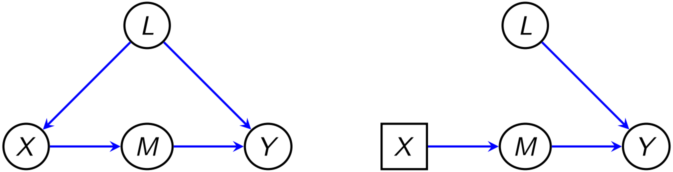

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$$p(L, M, Y \mid \mathrm{do}(X = \tilde{x})) = p(L) \quad \times \quad p(M \mid \tilde{x}) p(Y \mid L, M)$$

Intervention distributions (II)

Given a causal DAG $\mathcal{G}$ with distribution:

$$p(V) = \prod_{v \in V} p(v \mid \mathrm{pa}(v))$$

we wish to compute an intervention distribution via truncated factorization:

$$p(V \setminus X \mid \mathrm{do}(X = \mathbf{x})) \quad = \quad \prod_{v \in V \setminus X} p(v \mid \mathrm{pa}(v)).$$

Hence if we are interested in $Y \subset V \setminus X$ then we simply marginalize:

$$p(Y \mid \mathrm{do}(X = \mathbf{x})) \quad = \quad \sum_{w \in V \setminus (X \cup Y)} \quad \prod_{v \in V \setminus X} p(v \mid \mathrm{pa}(v)).$$

('g-computation' formula of Robins (1986); see also Spirtes *et al.* 1993.)

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Note: $p(Y \mid \text{do}(X = \mathbf{x}))$ is a sum over a product of terms $p(v \mid \text{pa}(v))$.

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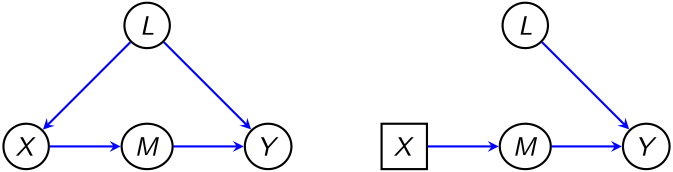

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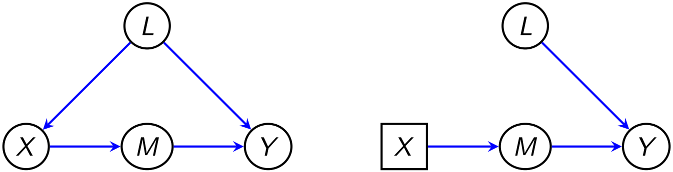

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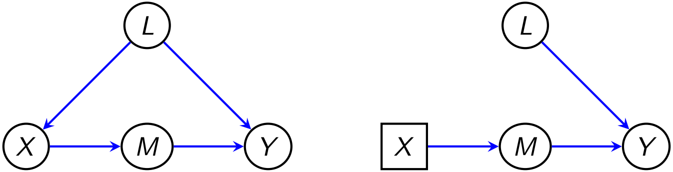

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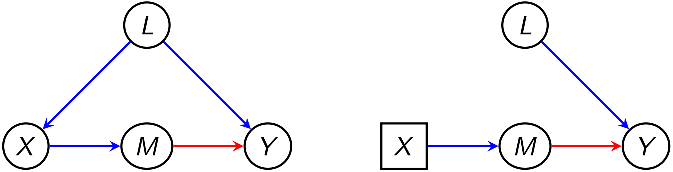
Note that $p(Y \mid \text{do}(X = \tilde{x})) \neq p(Y \mid X = \tilde{x})$.

Special case: no effect of M on Y



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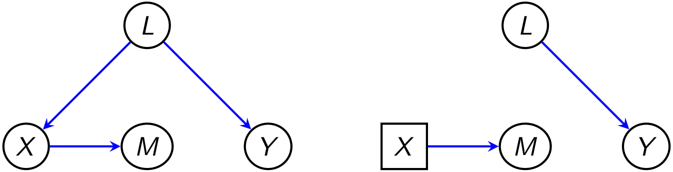

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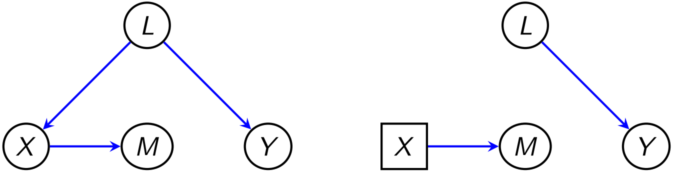

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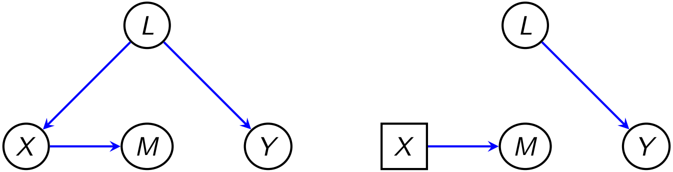

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$$p(Y \mid \text{do}(X = \tilde{x})) = \sum_{l,m} p(L = l)p(M = m \mid \tilde{x})p(Y \mid L = l)$$

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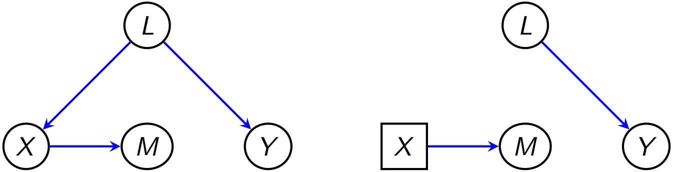

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$$\begin{aligned} p(Y \mid \mathrm{do}(X = \tilde{x})) &= \sum_{l,m} p(L = l)p(M = m \mid \tilde{x})p(Y \mid L = l) \\ &= \sum_{l} p(L = l)p(Y \mid L = l) \end{aligned}$$

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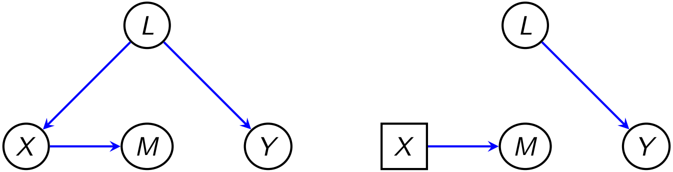

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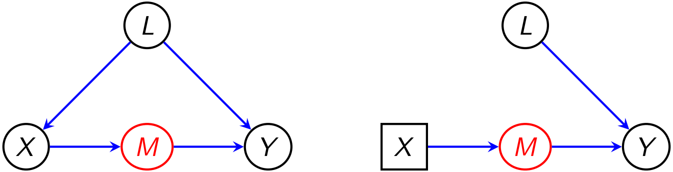
$$\begin{aligned} p(Y \mid \mathrm{do}(X = \tilde{x})) &= \sum_{l,m} p(L = l)p(M = m \mid \tilde{x})p(Y \mid L = l) \\ &= \sum_{l} p(L = l)p(Y \mid L = l) \\ &= p(Y) \neq P(Y \mid \tilde{x}) \end{aligned}$$

since $X \not\perp\!\!\!\perp Y$. 'Correlation is not Causation'.

Example with M unobserved



$$p(Y \mid \text{do}(X=\tilde{x})) = \sum_{l,m} p(L=l)p(M=m \mid \tilde{x})p(Y \mid L=l, M=m)$$

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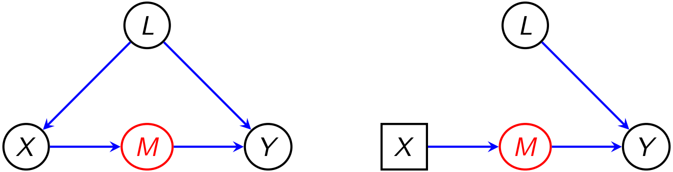
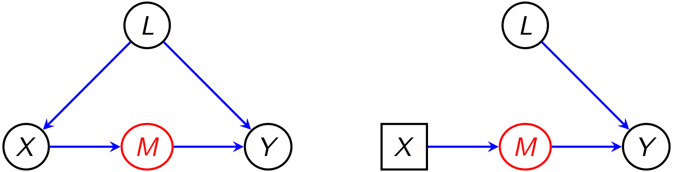


$$p(Y \mid \text{do}(X = \tilde{x})) = \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}) p(Y \mid L = l, M = m)$$

$$= \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}, L = l) p(Y \mid L = l, M = m, X = \tilde{x})$$

Here we have used that $M \perp\!\!\!\perp L \mid X$ and $Y \perp\!\!\!\perp X \mid L, M$.

Example with M unobserved



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p(Y \mid \mathrm{do}(X = \tilde{x})) &= \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}) p(Y \mid L = l, M = m) \\
&= \sum_{l,m} p(L = l) p(M = m \mid \tilde{x}, L = l) p(Y \mid L = l, M = m, X = \tilde{x}) \\
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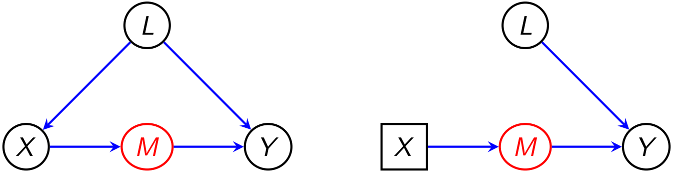

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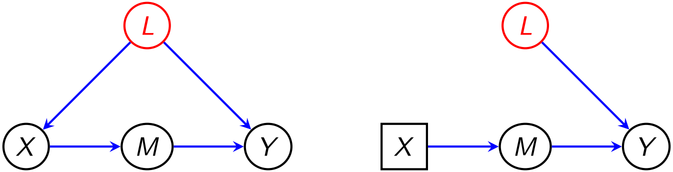
&= \sum_{l} p(L=l)p(Y \mid L=l, X=\tilde{x}).
\end{aligned}
$$

$\Rightarrow$ can find $p(Y \mid \mathrm{do}(X=\tilde{x}))$ even if M not observed.

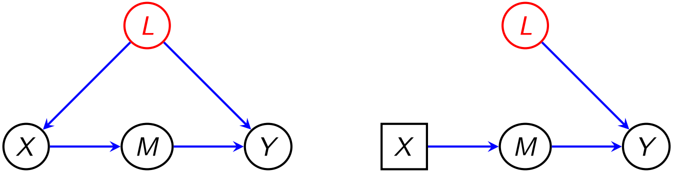
This is an example of the ‘back door formula’, aka ‘standardization’.

Example with L unobserved



$p(Y \mid \mathrm{do}(X = \tilde{x}))$

Example with L unobserved



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p(Y \mid \mathrm{do}(X = \tilde{x})) \\
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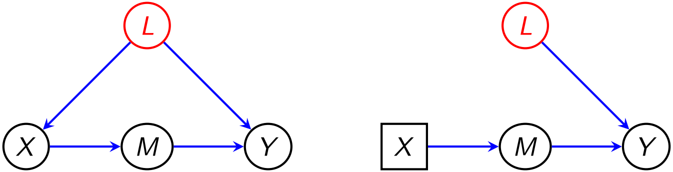
$$

Example with L unobserved



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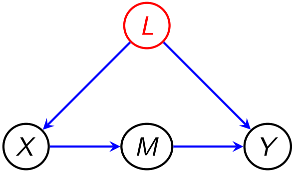

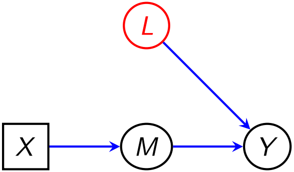

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&= \sum_m p(M = m \mid X = \tilde{x}) \left(\sum_{x^*} p(X = x^*)p(Y \mid M = m, X = x^*) \right)
\end{aligned}
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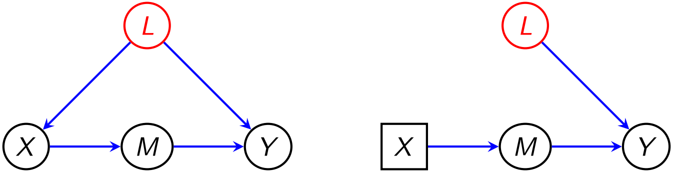
Example with L unobserved



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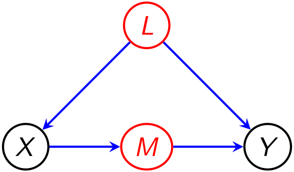
$\Rightarrow$ can find $p(Y \mid \mathrm{do}(X = \tilde{x}))$ even if L not observed.

This is an example of the 'front door formula' of Pearl (1995).

But with *both* L and M unobserved....



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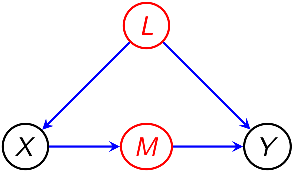
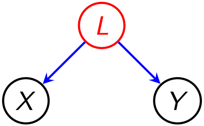


...we are out of luck!

Given $P(X, Y)$, absent further assumptions we cannot distinguish:



General Identification Question

Given: a latent DAG $\mathcal{G}(O \cup H)$, where O are observed, H are hidden, and disjoint subsets $X, Y \subseteq O$.

Q: Is $p(Y \mid \text{do}(X))$ identified given $p(O)$?

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or report that $p(Y \mid \text{do}(X))$ is not identified.

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Motivations:

- Characterize which interventions can be identified without parametric assumptions;
- Understand which functionals of the observed margin have a causal interpretation;

Latent Projection

Can preserve conditional independences and causal coherence with latents using paths. DAG $\mathcal{G}$ on vertices $V = O \dot{\cup} H$, define **latent projection** as follows: (Verma and Pearl, 1992)

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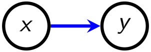

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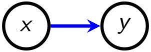

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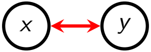

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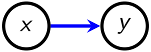

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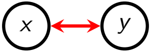
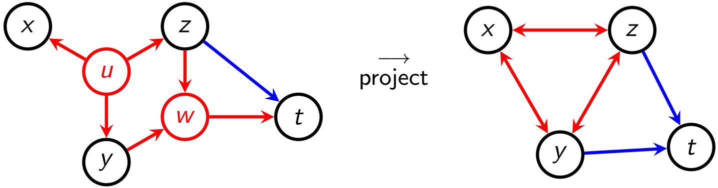
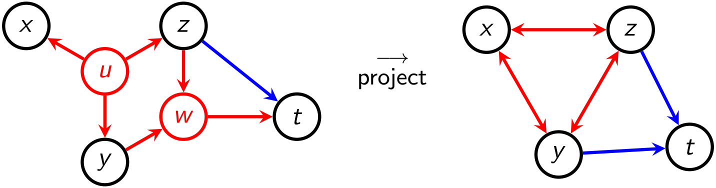


Then remove all latent variables H from the graph.

ADMGs



ADMGs



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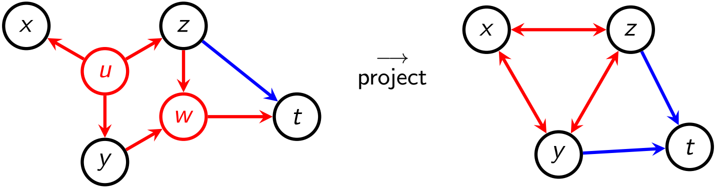
ADMGs



Latent projection leads to an **acyclic directed mixed graph** (ADMG)

Can read off independences with d/m-separation.

The projection preserves the causal structure; Verma and Pearl (1992).

‘Conditional’ Acyclic Directed Mixed Graphs

An ‘conditional’ acyclic directed mixed graph (CADMG) is a bi-partite graph $\mathcal{G}(V, W)$, used to represent structure of a distribution over V, indexed by W, for example $P(V \mid \mathrm{do}(W))$.

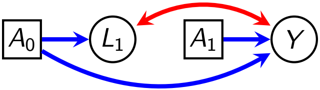
We require:

(i) The induced subgraph of $\mathcal{G}$ on V is an ADMG;

(ii) The induced subgraph of $\mathcal{G}$ on W contains no edges;

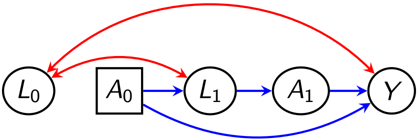
(iii) Edges between vertices in W and V take the form $w \rightarrow v$.

We represent V with circles, W with squares:



Here $V = \{L_1, Y\}$ and $W = \{A_0, A_1\}$.

Ancestors and Descendants



In a CADMG $\mathcal{G}(V, W)$ for $v \in V$, let the set of *ancestors* , *descendants* of v be:

$$\text{an}_{\mathcal{G}}(v) = \{a \mid a \to \cdots \to v \text{ or } a = v \text{ in } \mathcal{G}, a \in V \cup W\},$$

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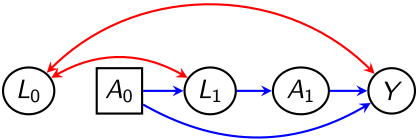
Ancestors and Descendants



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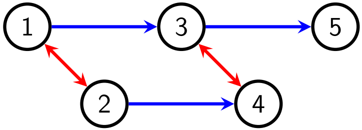
$$\text{an}(y) = \{a_0, l_1, a_1, y\}.$$

Districts

Define a **district** in a C/ADMG to be maximal sets connected by bi-directed edges:



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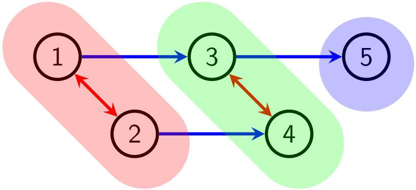

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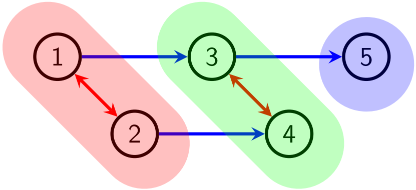

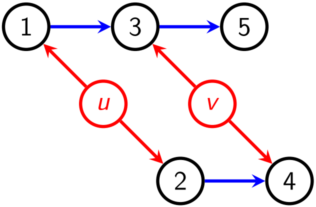

$$\sum_{u,v} p(u)\, p(x_1 \mid u)\, p(x_2 \mid u) \quad p(v)\, p(x_3 \mid x_1, v)\, p(x_4 \mid x_2, v) \quad p(x_5 \mid x_3)$$

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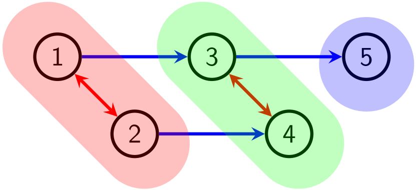

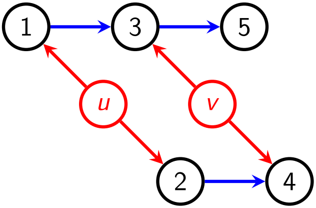

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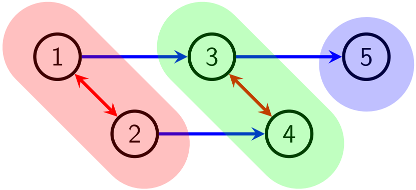

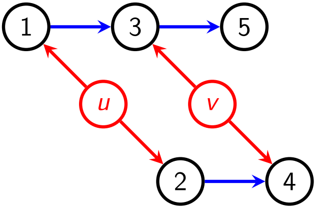

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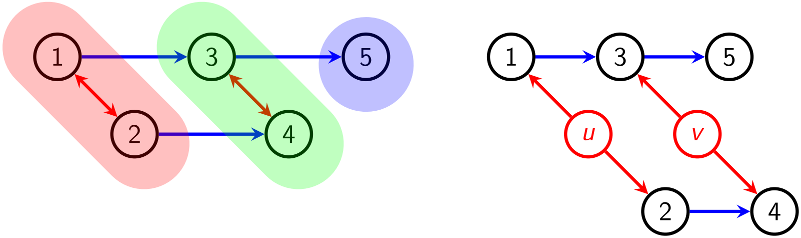

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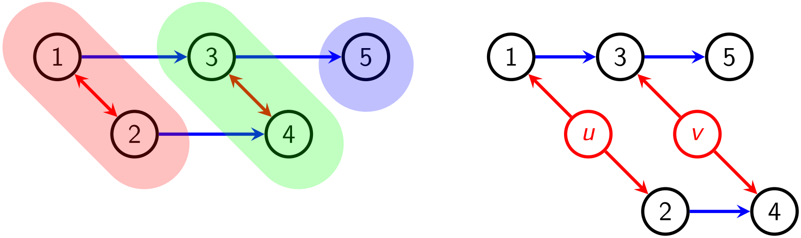

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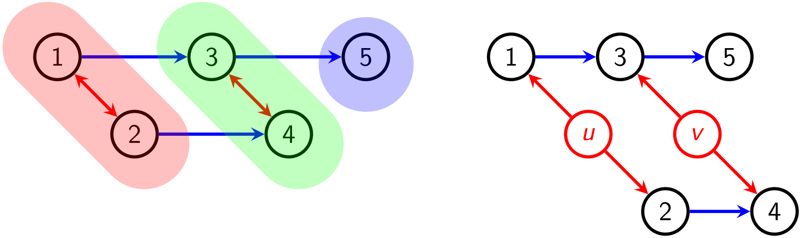

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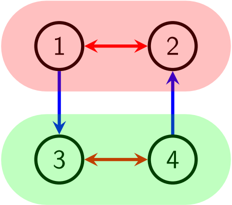
Districts are called 'c-components' by Tian.

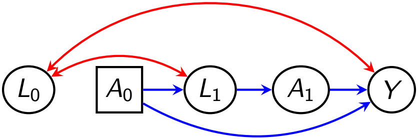
Edges between districts



There is no ordering on vertices such that parents of a district precede every vertex in the district.

(Cannot form a 'chain graph' ordering.)

Notation for Districts



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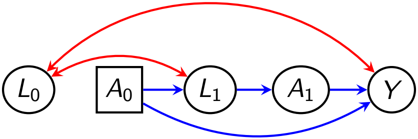

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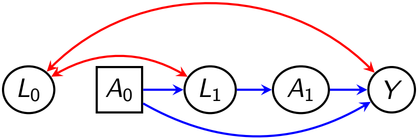

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We use $\mathcal{D}(\mathcal{G})$ to denote the set of districts in $\mathcal{G}$.

In example $\mathcal{D}(\mathcal{G}) = \{ \ \{l_0, l_1, y\}, \{a_1\} \ \}$.

Tian's ID algorithm for identifying $P(Y \mid \mathbf{do}(X))$

Jin Tian

(A) Re-express the query as a sum over a product of intervention distributions on districts:

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This is clearly sufficient for identifiability.

Necessity follows from results of Shpitser (2006); see also Huang and Valtorta (2006).

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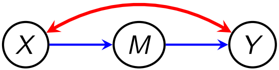
Then:

$$P(Y \mid \text{do}(X)) = \sum_{T \setminus (X \cup Y)} \prod_{D_i} p(D_i \mid \text{do}(\text{pa}(D_i) \setminus D_i)).$$

Example: front door graph

$\mathcal{G}$



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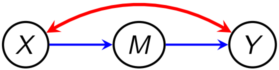

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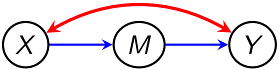

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Districts in $T \setminus \{X\}$ are $D_1 = \{M\}$, $D_2 = \{Y\}$.

$$p(Y \mid \mathrm{do}(X)) = \sum_M p(M \mid \mathrm{do}(X)) p(Y \mid \mathrm{do}(M))$$

Example: Sequentially randomized trial

A_1 is randomized; A_2 is randomized conditional on L, A_1;

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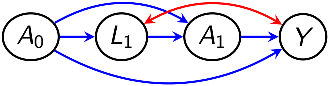

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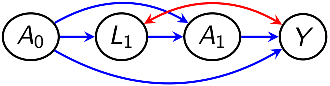

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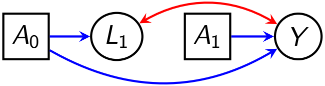
$\mathcal{G}_{[V \setminus \{A_0, A_1\}]}$



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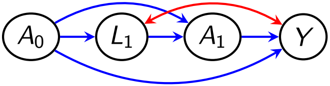

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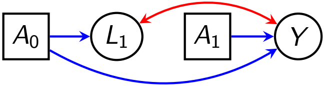

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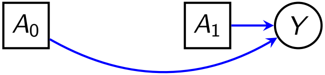

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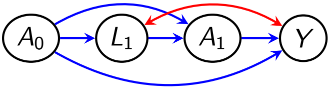

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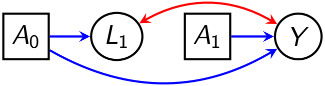

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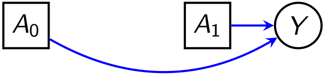

$D_1 = \{Y\}$

(Here the decomposition is trivial since there is only one district and no summation.)

(B) Finding if $P(D \mid \mathbf{do}(\mathrm{pa}(D) \setminus D))$ is identified

Idea: Find an ordering $r_1, \ldots, r_p$ of $O \setminus D$ such that:

If $P(O \setminus \{r_1, \ldots, r_{t-1}\} \mid \mathrm{do}(r_1, \ldots, r_{t-1}))$ is identified

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'fixing' differs formally from 'do'/cutting edges since the latter does not preserve identifiability in general.

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To do:

- Give a graphical characterization of 'fixability';
- Construct the identifying formula.

The set of fixable vertices

Given a CADMG $\mathcal{G}(V, W)$ we define the set of fixable vertices,

$$F(\mathcal{G}) \equiv \{v \mid v \in V, \text{dis}_{\mathcal{G}}(v) \cap \text{de}_{\mathcal{G}}(v) = \{v\}\}.$$

In words, a vertex $v \in V$ is fixable in $\mathcal{G}$ if there is no (proper) descendant of v that is in the same district as v in $\mathcal{G}$.

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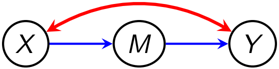
Thus v is fixable if there is no vertex $y \neq v$ such that

$$v \leftrightarrow \cdots \leftrightarrow y \quad \text{and} \quad v \rightarrow \cdots \rightarrow y \quad \text{in } \mathcal{G}.$$

Note that the set of fixable vertices is a subset of V, and contains at least one vertex from each district in $\mathcal{G}$.

Example: Front door graph

$\mathcal{G}$

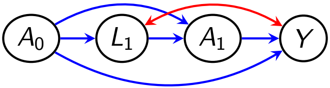


$F(\mathcal{G}) = \{M, Y\}$

X is not fixable since Y is a descendant of X and

Y is in the same district as X

Example: Sequentially randomized trial



Here $F(\mathcal{G}) = \{A_0, A_1, Y\}$.

L_1 is not fixable since Y is a descendant of L_1 and

Y is in the same district as L_1.

The *graphical* operation of fixing vertices

Given a CADMG $\mathcal{G}(V, W, E)$, for every $r \in F(\mathcal{G})$ we associate a transformation ϕ_r on the pair $(\mathcal{G}, P(X_V \mid X_W))$:

$$\phi_r(\mathcal{G}) \equiv \mathcal{G}^\dagger(V \setminus \{r\}, W \cup \{r\}),$$

where in $\mathcal{G}^\dagger$ we remove from $\mathcal{G}$ any edge that has an arrowhead at r.

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The operation of 'fixing r' simply transfers r from 'V' to 'W', and removes edges $r \leftrightarrow$ or $r \leftarrow$.

Example: front door graph

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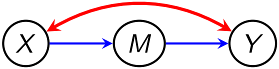

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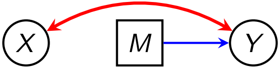
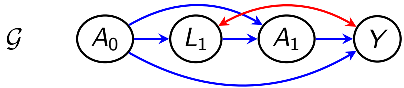


$F(\phi_M(\mathcal{G})) = \{X, Y\}$

Note that X was not fixable in $\mathcal{G}$,

but it is fixable in $\phi_M(\mathcal{G})$ after fixing M.

Example: Sequentially randomized trial



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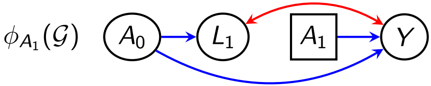

Notice $F(\phi_{A_1}(\mathcal{G})) = \{A_0, L_1, Y\}$.

Thus L_1 was not fixable prior to fixing A_1,

but L_1 is fixable in $\phi_{A_1}(\mathcal{G})$ after fixing A_1.

The *probabilistic* operation of fixing vertices

Given a distribution $P(V \mid W)$ we associate a transformation:

$$\phi_r(P(V \mid W); \mathcal{G}) \equiv \frac{P(V \mid W)}{P(r \mid \mathrm{mb}_{\mathcal{G}}(r))}.$$

Here
$\mathrm{mb}_{\mathcal{G}}(r) = \{y \neq r \mid (r \leftarrow y) \text{ or } (r \leftrightarrow \circ \cdots \circ \leftrightarrow y) \text{ or } (r \leftrightarrow \circ \cdots \circ \leftrightarrow \circ \leftarrow y)\}.$

In words: *we divide by the conditional distribution of r given the other vertices in the district containing r, and the parents of the vertices in that district.*

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In words: *we divide by the conditional distribution of r given the other vertices in the district containing r, and the parents of the vertices in that district.*

It can be shown that if r is fixable in $\mathcal{G}$ then:

$$\phi_r(P(V \mid \text{do}(W)); \mathcal{G}) = P(V \setminus \{r\} \mid \text{do}(W \cup \{r\})).$$

as required.

Note: If r is fixable in $\mathcal{G}$ then $\text{mb}_{\mathcal{G}}(r)$ is the 'Markov blanket' of r in $\text{an}_{\mathcal{G}}(\text{dis}_{\mathcal{G}}(r))$.

Unifying Marginalizing and Conditioning

Some special cases:

- If $\text{mb}_{\mathcal{G}}(r) = (V \cup W) \setminus \{r\}$ then fixing corresponds to marginalizing:

$$\phi_r(P(V \mid W); \mathcal{G}) = \frac{P(V \mid W)}{P(r \mid (V \cup W) \setminus \{r\})} = P(V \setminus \{r\} \mid W)$$

- If $\text{mb}_{\mathcal{G}}(r) = W$ then fixing corresponds to ordinary conditioning:

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- In the general case fixing corresponds to re-weighting, so

$$\phi_r(P(V \mid W); \mathcal{G}) = P^*(V \setminus \{r\} \mid W \cup \{r\}) \neq P(V \setminus \{r\} \mid W \cup \{r\})$$

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Having a single operation simplifies the identification algorithm.

Composition of fixing operations

We use $\circ$ to indicate composition of operations in the natural way.

If s is fixable in $\mathcal{G}$ and then r is fixable in $\phi_s(\mathcal{G})$ (after fixing s) then:

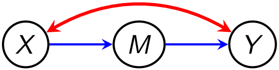
$$\phi_r \circ \phi_s(\mathcal{G}) \equiv \phi_r(\phi_s(\mathcal{G}))$$

$$\phi_r \circ \phi_s(P(V \mid W); \mathcal{G}) \equiv \phi_r\left(\phi_s\left(P(V \mid W); \mathcal{G}\right); \phi_s(\mathcal{G})\right)$$

Back to step (B) of identification

Recall our goal is to identify $P(D \mid \text{do}(\text{pa}(D) \setminus D))$, for the districts D in $\mathcal{G}^*$:

$\mathcal{G}$



$p(Y \mid \text{do}(X))$

Districts in $T \setminus \{X\}$ are $D_1 = \{M\}$, $D_2 = \{Y\}$.

$$p(Y \mid \text{do}(X)) = \sum_M p(M \mid \text{do}(X)) p(Y \mid \text{do}(M))$$

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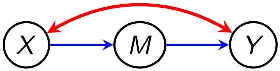

$p(Y \mid \text{do}(X))$

$\mathcal{G}_{[V \setminus \{X\}]} = \mathcal{G}^*$

$T = \{X, M, Y\}$

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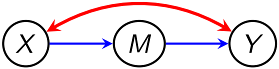
Example: front door graph: $D_1 = \{M\}$

$\mathcal{G}$



$F(\mathcal{G}) = \{M, Y\}$

$\phi_Y(\mathcal{G})$

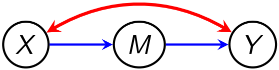
$F(\phi_Y(\mathcal{G})) = \{X, M\}$

$\phi_X \circ \phi_Y(\mathcal{G})$

This proves that $p(M \mid \text{do}(X))$ is identified.

Example: front door graph: $D_2 = \{Y\}$

$\mathcal{G}$



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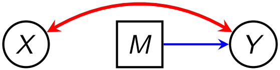

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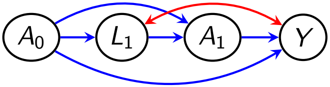
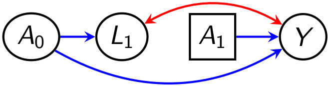
Example: Sequential Randomization

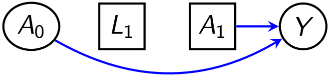
$\mathcal{G}$



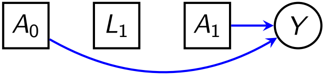
$\phi_{A_1}(\mathcal{G})$



$\phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$



$\phi_{A_0} \circ \phi_{L_1} \circ \phi_{A_1}(\mathcal{G})$



This establishes that $P(Y \mid \text{do}(A_0, A_1))$ is identified.

Review: Tian's ID algorithm via fixing

(A) Re-express the query as a sum over a product of intervention distributions on districts:

$$p(Y \mid \text{do}(X)) = \sum \prod_i p(D_i \mid \text{do}(\text{pa}(D_i) \setminus D_i)).$$

- Cut edges into X;
- Restrict to vertices that are (still) ancestors of Y;
- Find the set of districts $D_1, \dots, D_p$.

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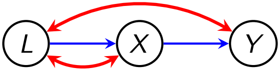
(B) Check whether each term: $p(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i))$ is identified:

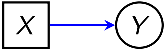
- Iteratively find a vertex that r_t that is fixable in $\phi_{r_{t-1}} \circ \cdots \circ \phi_{r_1}(\mathcal{G})$, with $r_t \notin D_i$;
- If no such vertex exists then $P(D_i \mid \mathrm{do}(\mathrm{pa}(D_i) \setminus D_i))$ is not identified.

Not identified example

$\mathcal{G}$



$\mathcal{G}^*$



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There is one district $D = \{Y\}$ in $\mathcal{G}^*$.

Not identified example

Suppose we wish to find $p(Y \mid \text{do}(X))$.

There is one district $D = \{Y\}$ in $\mathcal{G}^*$.

But since the only fixable vertex in $\mathcal{G}$ is Y, we see that $p(Y \mid \text{do}(X))$ is not identified.

Reachable subgraphs of an ADMG

A CADMG $\mathcal{G}(V, W)$ is *reachable* from ADMG $\mathcal{G}^*(V \cup W)$ if there is an ordering of the vertices in $W = \langle w_1, \ldots, w_k \rangle$, such that for $j = 1, \ldots, k$,

$$w_1 \in F(\mathcal{G}^*) \text{ and for } j = 2, \ldots, k,$$
$$w_j \in F(\phi_{w_{j-1}} \circ \cdots \circ \phi_{w_1}(\mathcal{G}^*)).$$

Thus a subgraph is reachable if, under some ordering, each of the vertices in W may be fixed, first in $\mathcal{G}^*$, and then in $\phi_{w_1}(\mathcal{G}^*)$, then in $\phi_{w_2}(\phi_{w_1}(\mathcal{G}^*))$, and so on.

Invariance to orderings

In general, there may exist multiple sequences that fix a set W, however, they all result in both the same graph and distribution.

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This is a consequence of the following:

Lemma

Let $\mathcal{G}(V, W)$ be a CADMG with $r, s \in \mathbb{F}(\mathcal{G})$, and let $q_V(V \mid W)$ be Markov w.r.t. $\mathcal{G}$, and further (a) $\phi_r(q_V; \mathcal{G})$ is Markov w.r.t. $\phi_r(\mathcal{G})$; and (b) $\phi_s(q_V; \mathcal{G})$ is Markov w.r.t. $\phi_s(\mathcal{G})$. Then

$$\begin{aligned} \phi_r \circ \phi_s(\mathcal{G}) &= \phi_s \circ \phi_r(\mathcal{G}); \\ \phi_r \circ \phi_s(q_V; \mathcal{G}) &= \phi_s \circ \phi_r(q_V; \mathcal{G}). \end{aligned}$$

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Consequently, if $\mathcal{G}(V, W)$ is reachable from $\mathcal{G}(V \cup W)$ then $\phi_V(p(V, W); \mathcal{G})$ is uniquely defined.

Intrinsic sets

A set D is said to be *intrinsic* if it forms a *district* in a *reachable* subgraph. If D is intrinsic in $\mathcal{G}$ then $p(D \mid \text{do}(\text{pa}(D) \setminus D))$ is identified.

Let $\mathcal{I}(\mathcal{G})$ denote the intrinsic sets in $\mathcal{G}$.

Theorem

Let $\mathcal{G}(H \cup V)$ be a causal DAG with latent projection $\mathcal{G}(V)$. For $A \dot{\cup} Y \subset V$, let $Y^* = \text{an}_{\mathcal{G}(V)_{V \setminus A}}(Y)$. Then if $\mathcal{D}(\mathcal{G}(V)_{Y^*}) \subseteq \mathcal{I}(\mathcal{G}(V))$,

$$p(Y \mid \text{do}_{\mathcal{G}(H \cup V)}(A)) = \sum_{Y^* \setminus Y} \prod_{D \in \mathcal{D}(\mathcal{G}(V)_{Y^*})} \phi_{V \setminus D}(p(V); \mathcal{G}(V)). \qquad (*)$$

If not, there exists $D \in \mathcal{D}(\mathcal{G}(V)_{Y^*}) \setminus \mathcal{I}(\mathcal{G}(V))$ and $p(Y \mid \text{do}_{\mathcal{G}(H \cup V)}(A))$ is not identifiable in $\mathcal{G}(H \cup V)$.

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Shpitser+R+Robins (2012) give an efficient algorithm for computing $(*)$.

Part Two: The Nested Markov Model

1. Motivation

2. Deriving constraints via fixing

3. The Nested Markov Model

4. Finer Factorizations

5. Discrete Parameterization

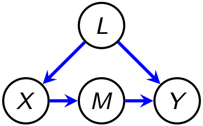
6. Testing and Fitting

7. Completeness

Outline

Motivation

- So far we have shown how to estimate interventional distributions separately, but no guarantee these estimates are coherent.
- We also may have multiple identifying expressions: which one should we use?



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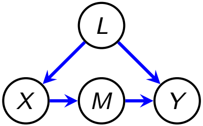

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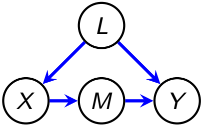

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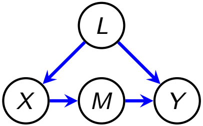

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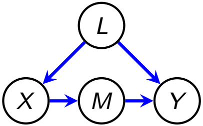

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- Even better, if model can be shown smooth we get nice asymptotics for free.

All this suggests we should define a model which we can parameterize.

Outline

Deriving constraints via fixing

Let $p(O)$ be the observed margin from a DAG with latents $\mathcal{G}(O \cup H)$,

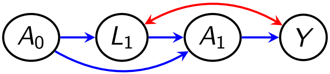
Idea: If $r \in O$ is fixable then $\phi_r(p(O); \mathcal{G})$ will obey the Markov property for the graph $\phi_r(\mathcal{G})$.

...and this can be iterated.

This gives non-parametric constraints that are not independences, that are implied by the latent DAG.

Example: The 'Verma' Constraint

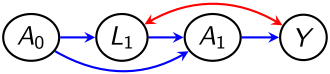
$\mathcal{G}$



This graph implies no conditional independences on $P(A_0, L_1, A_1, Y)$.

Example: The 'Verma' Constraint

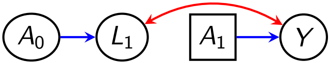
$\mathcal{G}$



This graph implies no conditional independences on $P(A_0, L_1, A_1, Y)$.

But since $F(\mathcal{G}) = \{A_0, A_1, Y\}$, we may construct:

$\phi_{A_1}(\mathcal{G})$



$$\phi_{A_1}(p(A_0, L_1, A_1, Y)) \quad = \quad p(A_0, L_1, A_1, Y)/p(A_1 \mid A_0, L_1)$$

$$A_0 \perp\!\!\!\perp Y \mid A_1 \qquad [\phi_{A_1}(p(A_0, L_1, A_1, Y); \mathcal{G})]$$

Outline

The nested Markov model

These independences may be used to define a graphical model:

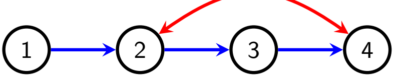
Definition

$p(V)$ obeys the *global nested Markov property* for $\mathcal{G}$ if for all reachable sets R, the kernel $\phi_{V\setminus R}(p(V);\mathcal{G})$ obeys the global Markov property for $\phi_{V\setminus R}(\mathcal{G})$.

This is a 'generalized' Markov property since it is defined by conditional independence in re-weighted distributions (obtained via fixing).

We will use $\mathcal{N}(\mathcal{G})$ to indicate the set of distributions obeying this property.

Notation



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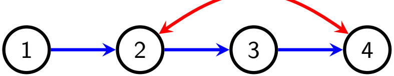

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Theorem (R,Evans, Shpitser, Robins, 2017)

For a positive distribution $p \in \mathcal{N}(\mathcal{G})$ and vertices v_1, v_2 fixable in $\mathcal{G}$,

$$(\phi_{v_1} \circ \phi_{v_2})(p) = (\phi_{v_2} \circ \phi_{v_1})(p).$$

Hence, the order of fixing doesn't matter.

This is another way of saying that all identifying expressions for a causal quantity will be the same.

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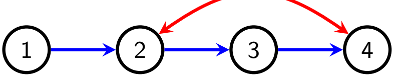

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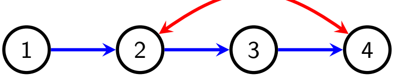

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For any reachable R this justifies the (unambiguous) notation $\phi_{V \setminus R}$.

For $p \in \mathcal{N}(\mathcal{G})$, let

$$\mathcal{G}[R] \equiv \phi_{V \setminus R}(\mathcal{G}) \qquad\qquad q_R \equiv \phi_{V \setminus R}(p).$$

be respectively, the graph and distribution where $V \setminus R$ is fixed.

Reachable CADMGs

Note that $\mathcal{G}[R]$ is always just the CADMG with:

- random vertices R,
- fixed vertices $\text{pa}_{\mathcal{G}}(R) \setminus R$,
- induced edges from $\mathcal{G}$ among R and of the form $\text{pa}_{\mathcal{G}}(R) \to R$.

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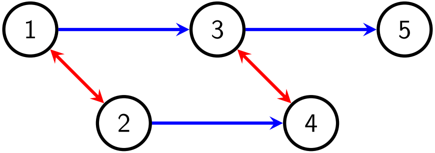

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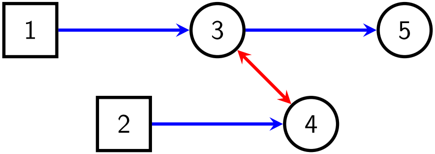

Graph shown is $\mathcal{G}[\{3, 4, 5\}]$.

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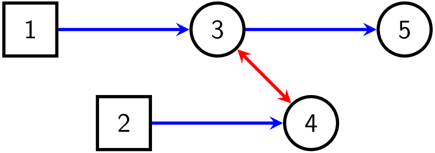
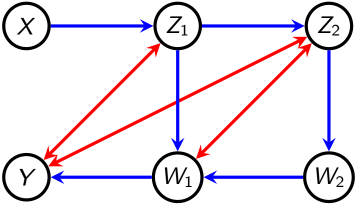


Graph shown is $\mathcal{G}[\{3, 4, 5\}]$.

Also recall that if there is an underlying causal DAG then $p(x_V)$ then:

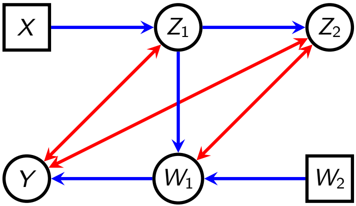
$$q_R(x_R \mid x_{\text{pa}(R)\setminus R}) = p(x_R \mid do(x_{V\setminus R})).$$

Example



$p(x, y, w_1, w_2, z_1, z_2)$

Example



$$q_{yw_1z_1z_2}(y, w_1, z_1, z_2 \mid x, w_2) = \frac{p(x, y, w_1, w_2, z_1, z_2)}{p(x)p(w_2 \mid z_2)}$$

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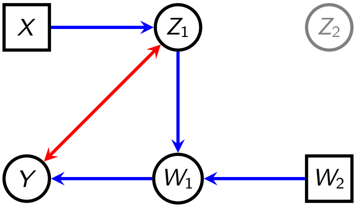

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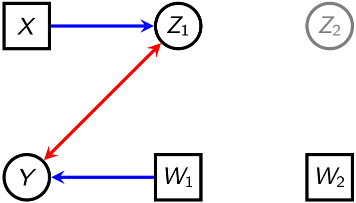

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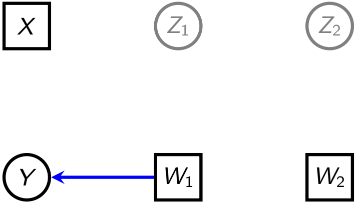

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and $q_{yz_1}(y \mid x, w_1)$ doesn't depend upon x.

Nested Markov Model

Various equivalent formulations:

Factorization into Districts.
For each reachable R in $\mathcal{G}$,

$$q_R(x_R \,|\, x_{\mathrm{pa}(R)\setminus R}) = \prod_{D \in \mathcal{D}(\mathcal{G}[R])} f_D(x_{D \cup \mathrm{pa}(D)})$$

some functions f_D.

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For each reachable R in $\mathcal{G}$,

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Ordered Local Markov Property.
For every intrinsic S and v maximal in S under some topological ordering,

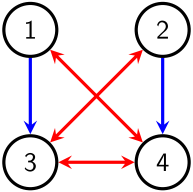
$$X_v \perp\!\!\!\perp X_{V \setminus \mathrm{mb}_{\mathcal{G}[S]}(v)} \,|\, X_{\mathrm{mb}_{\mathcal{G}[S]}(v)} \, [q_S].$$

Theorem. These are all equivalent.

Outline

Heads and Tails

As established, we can factorize a graph into districts; however, finer factorizations are possible.



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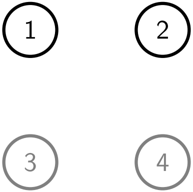

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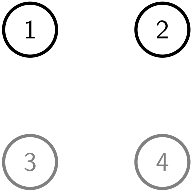

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So could factorize this as

$$\begin{aligned} p(x_1, x_2, x_3, x_4) &= p(x_1, x_2) p(x_3, x_4 \mid x_1, x_2) \\ &= p(x_1) p(x_2) p(x_3, x_4 \mid x_1, x_2). \end{aligned}$$

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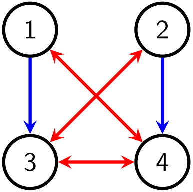

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Note that the vertices $\{3, 4\}$ can't be d-separated from one another.

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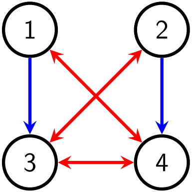

But vertices in $S \setminus H$ may factorize:

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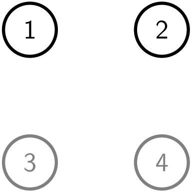

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Factorizations

Recursively define a partition of reachable sets as follows. If R has multiple districts,

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Theorem (Head Factorization Property)

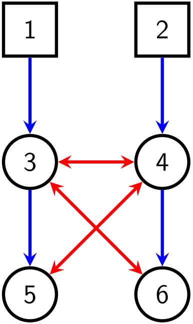
p obeys the nested Markov property for $\mathcal{G}$ if and only if for every reachable set R,

$$q_R(x_R \,|\, x_{\text{pa}(R)\setminus R}) = \prod_{H \in [R]_{\mathcal{G}}} q_H(x_H \,|\, x_T).$$

Here $q_H \equiv q_{S(H)}$ is density associated with intrinsic set for H.
(Recursive heads are in one-to-one correspondence with intrinsic sets.)

Heads and Tails

Recall, intrinsic sets are reachable districts:



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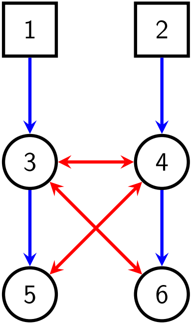

intrinsic set	I	$\{3, 4, 5, 6\}$
recursive head	H	$\{5, 6\}$
tail	T	$\{1, 2, 3, 4\}$

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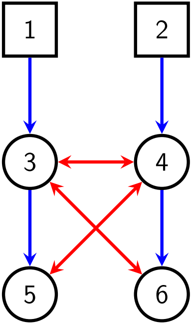

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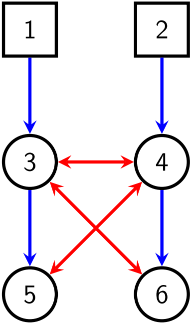

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So

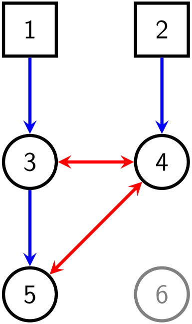
$$[\{3,4,5,6\}]_{\mathcal{G}} = \{\{3,4\},\{5,6\}\}.$$

Factorization:

$$q_{3456}(x_{3456} \mid x_{12}) = q_{56}(x_{56} \mid x_{1234}) \cdot q_{34}(x_{34} \mid x_{12})$$

Heads and Tails

What if we fix 6 first?



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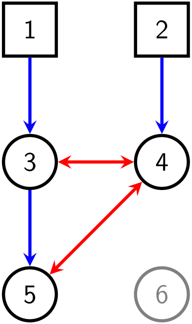

intrinsic set	I	$\{3, 4, 5\}$
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tail	T	$\{1, 2, 3\}$

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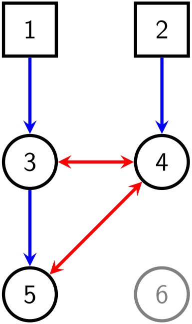

intrinsic set	I	$\{3, 4, 5\}$
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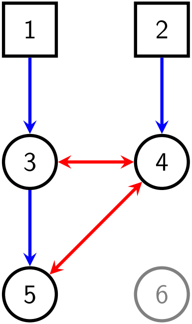

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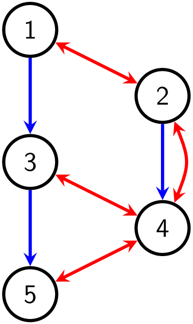
So

$$[\{3,4,5\}]_{\mathcal{G}} = \{\{3\},\{4,5\}\}.$$

Factorization:

$$q_{345}(x_{345} \mid x_{12}) = q_{45}(x_{45} \mid x_{123}) \cdot q_3(x_3 \mid x_1)$$

Heads and Tails



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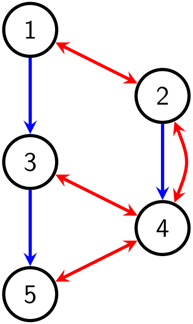

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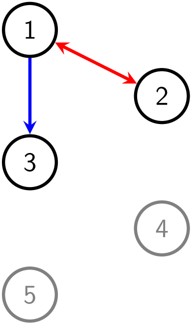
tail	T	$\{1,2,3\}$

Heads and Tails



intrinsic set	I	$\{1,2,3,4,5\}$
recursive head	H	$\{4,5\}$
tail	T	$\{1,2,3\}$

intrinsic set	I	$\{1,2\}$
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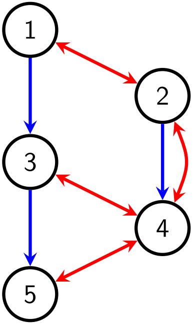

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Factorization:

$$q_{12345}(x_{12345}) = q_{45}(x_{45} \mid x_{123}) \cdot q_3(x_3 \mid x_1) \cdot q_{12}(x_{12}).$$

Outline

Parameterizations

Let $\mathcal{M}$ be a model (i.e. collection of probability distributions).

A **parameterization** is a continuous bijective map

$$\theta : \mathcal{M} \to \Theta$$

with continuous inverse, where Θ is an open subset of $\mathbb{R}^d$.

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We will assume all variables are binary; this extends easily to the general categorical / discrete case.

Parameterization

Say binary distribution p *parameterized according to* $\mathcal{G}$ if[1]

$$p(x_V \mid x_W) = \sum_{O \subseteq C \subseteq V} (-1)^{|C \setminus O|} \prod_{H \in [C]_{\mathcal{G}}} \theta_H(x_T),$$

for some parameters $\theta_H(x_T)$ where $O = \{v : x_v = 0\}$.

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Theorem (Evans and Richardson, 2015)

p is parameterized according to $\mathcal{G}$ if and only if it recursively factorizes according to $\mathcal{G}$ (so $p \in \mathcal{N}(\mathcal{G})$).

---

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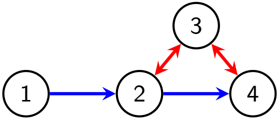
Probabilities



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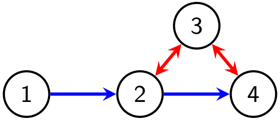

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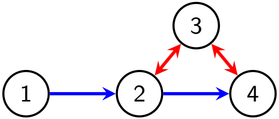

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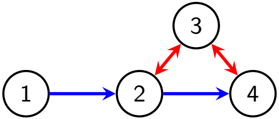

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For the district $\{2, 3, 4\}$ get

$$\begin{aligned} &q_{234}(0_2, 1_3, 1_4 \mid x_1) \\ &= q_{234}(0_2 \mid x_1) - q_{234}(0_{23} \mid x_1) - q_{234}(0_{24} \mid x_1) + q_{234}(0_{234} \mid x_1) \end{aligned}$$

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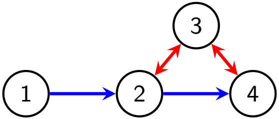

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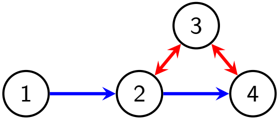

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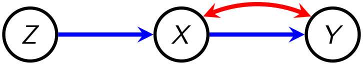
Putting this all together:

$$\begin{aligned} &p(1_1, 0_2, 1_3, 1_4) \\ &= \{1 - \theta_1\} \{\theta_2(1) - \theta_{23}(1) - \theta_2(1)\theta_4(0) + \theta_2(1)\theta_{34}(1, 0)\}. \end{aligned}$$

Example 1



Intrinsic Sets	Z	X, Y	X

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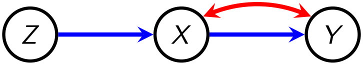

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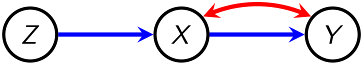

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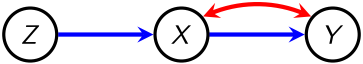

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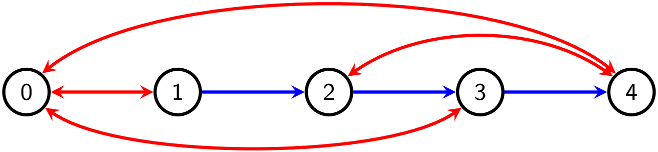
So parameterization is just

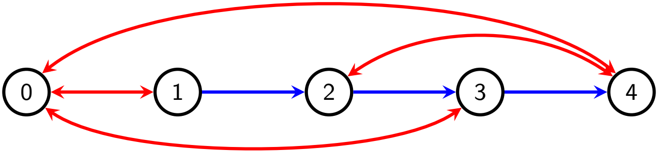
$$p(z = 0), \qquad p(x = 0 \mid z) \qquad p(y = 0 \mid x, z).$$

Model is saturated.

Example 2



Example 2



$$p(0_0, 1_1, 1_2, 0_3, 0_4) = p(0_0, 1_1, 1_2, 0_3) \cdot q_4(0_4 \mid 0_0, 1_1, 1_2, 0_3)$$

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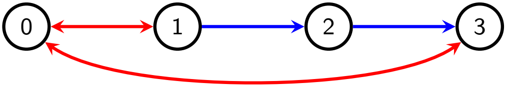

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&= \theta_{03}(1) - \theta_{013}(1)
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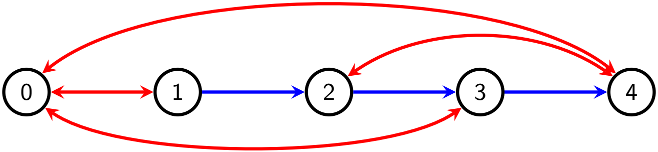

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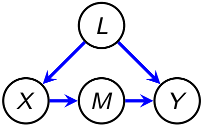
so

$$
p(0_0, 1_1, 1_2, 0_3, 0_4) = \{1 - \theta_2(1)\} \{\theta_{03}(1) - \theta_{013}(1)\} \cdot \theta_4(0, 1, 1, 0).
$$

Motivation

- So far we have shown how to estimate interventional distributions separately, but no guarantee these estimates are coherent.
- We also may have multiple identifying expressions: which one should we use?



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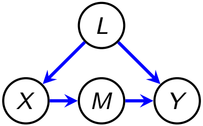

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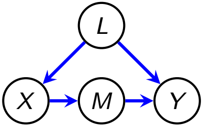

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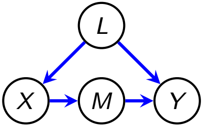

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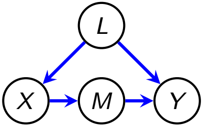

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All this suggests we should define a model which we can parameterize.

Outline

Exponential Families

Theorem

Let $\mathcal{N}(\mathcal{G})$ be the collection of binary distributions that recursively factorize according to $\mathcal{G}$. Then $\mathcal{N}(\mathcal{G})$ is a curved exponential family of dimension

$$d(\mathcal{G}) = \sum_{H \in \mathcal{H}(\mathcal{G})} 2^{|\operatorname{tail}(H)|}.$$

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(Shpitser et al., 2013) give an alternative log-linear parametrization.

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Currently no equivalent of PC algorithm for full nested model.

Can use FCI algorithm (Spirtes at al., 2000) for **ordinary Markov models** associated with ADMG (conditional independences only), in general this is a supermodel of the nested model (see Evans and Richardson, 2014).

Open Problems:

- Nested Markov equivalence;
- Constraint based search;
- Gaussian parametrization.

Outline

Completeness

Could the nested Markov property be further refined?

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However, there are additional **inequality constraints**. e.g. Instrumental inequalities, CHSH inequalities etc.,

Potentially unsatisfactory as may not be a causal model corresponding to our inferred parameters.

---

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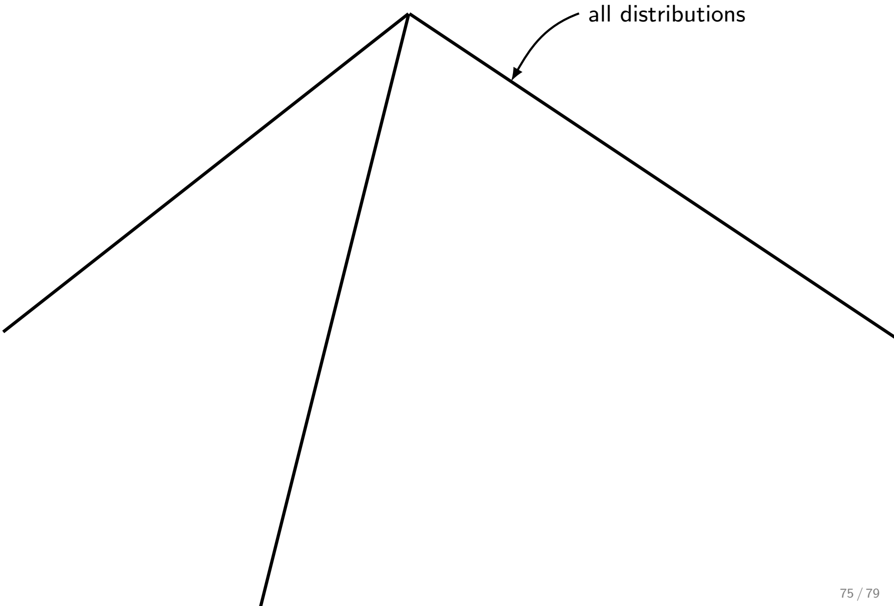
Big Picture



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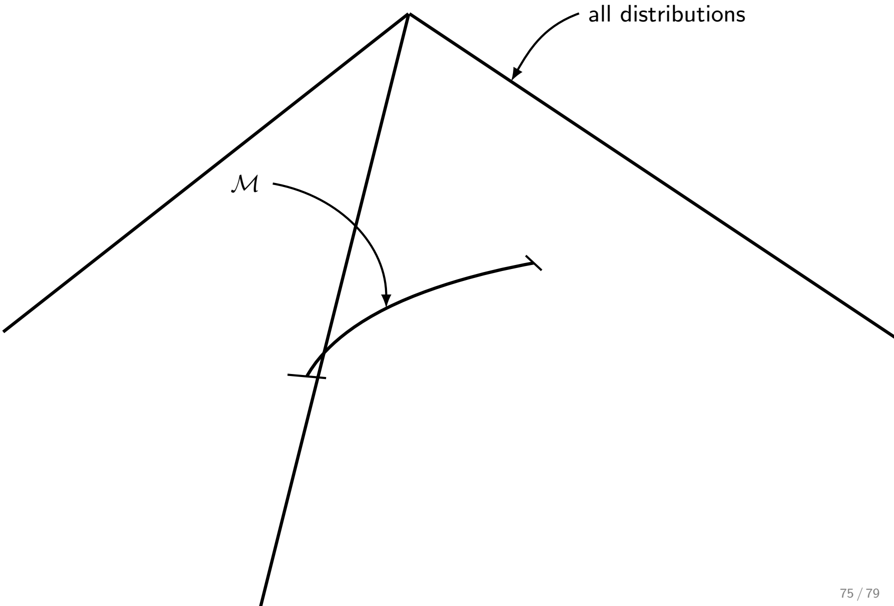

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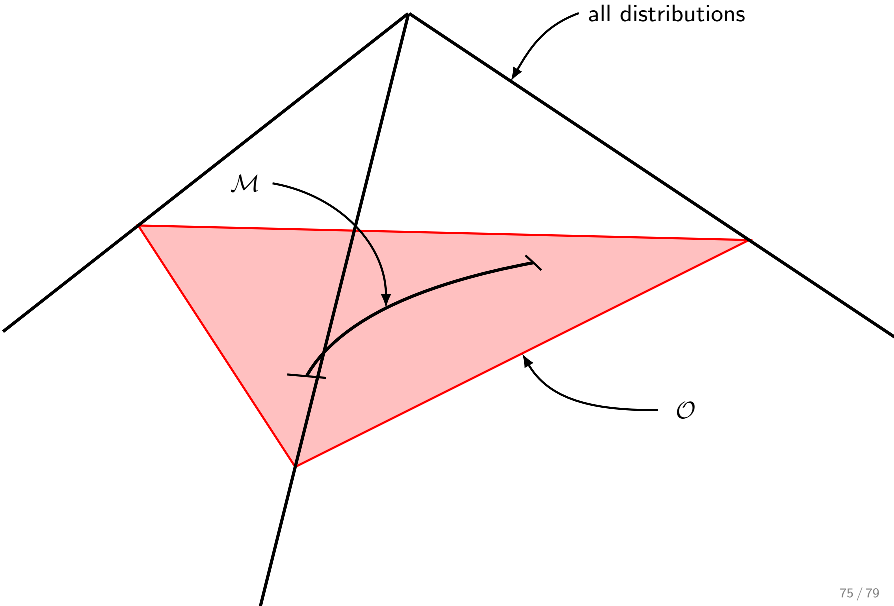

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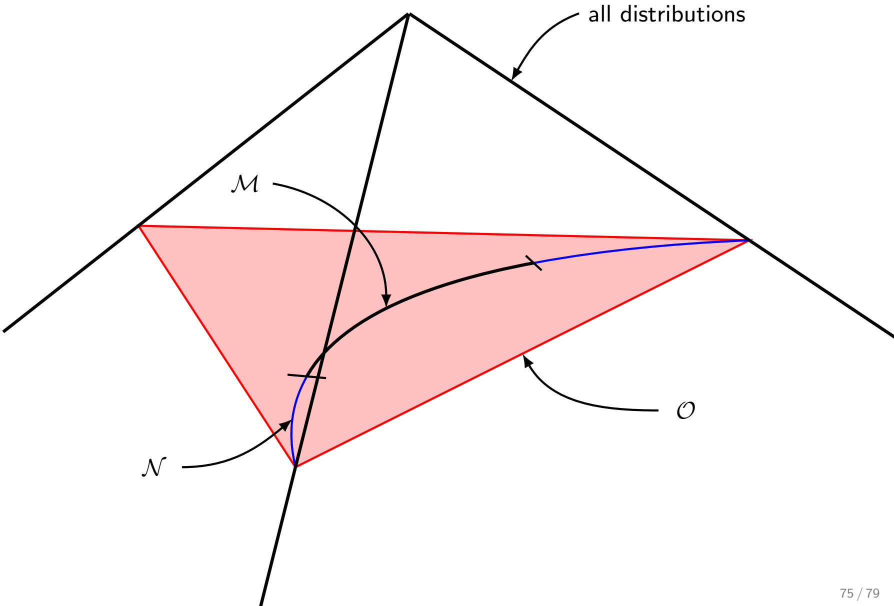

More on the nested Markov model

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- The nested model may also be defined by constraints resulting from an algorithm given in (Tian, 2002b).

Future Work

- Characterizing nested Markov equivalence;
- Methods for inferring graph structure.

Nested Markov model references

- Evans, R. J. (2015). Margins of discrete Bayesian networks. arXiv preprint:1501.02103.
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Partition Function for General Sets

Let $\mathcal{I}(\mathcal{G})$ be the intrinsic sets of $\mathcal{G}$. Define a partial ordering $\prec$ on $\mathcal{I}(\mathcal{G})$ by $S_1 \prec S_2$ if and only if $S_1 \subset S_2$. This induces an isomorphic partial ordering on the corresponding recursive heads.

For any $B \subseteq V$ let

$$\Phi_{\mathcal{G}}(B) = \{H \subseteq B \mid H \text{ maximal under } \prec \text{ among heads contained in } B\};$$
$$\phi_{\mathcal{G}}(B) = \bigcup_{H \in \Phi_{\mathcal{G}}(B)} H.$$

So $\Phi_{\mathcal{G}}(B)$ is the 'maximal heads' in B, $\phi_{\mathcal{G}}(B)$ is their union.

Define (recursively)

$$[\emptyset]_{\mathcal{G}} \equiv \emptyset$$
$$[B]_{\mathcal{G}} \equiv \Phi_{\mathcal{G}}(B) \cup [\phi_{\mathcal{G}}(B)]_{\mathcal{G}}.$$

Then $[B]_{\mathcal{G}}$ is a partition of B.

d-Separation

A **path** is a sequence of edges in the graph; vertices may not be repeated.

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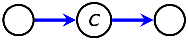
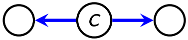
A path from v to w is **blocked** by $C \subseteq V \setminus \{v, w\}$ if either

(i) any non-collider is in C:





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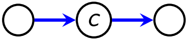
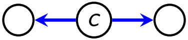
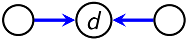
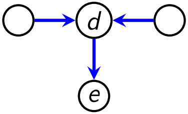




(ii) or any collider is not in C, nor has descendants in C:





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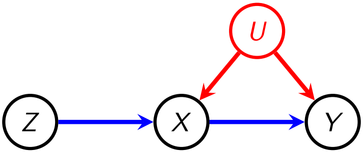
Two vertices v and w are **d-separated** given $C \subseteq V \setminus \{v, w\}$ if **all** paths are blocked.

The IV Model

Assume four variable DAG shown, but U unobserved.



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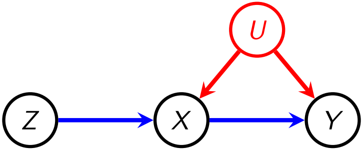

Marginalized DAG model

$$p(z, x, y) = \int p(u)\, p(z)\, p(x \mid z, u)\, p(y \mid x, u)\, du$$

Assume all observed variables are discrete; no assumption made about latent variables.

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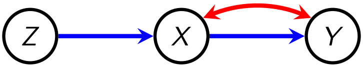

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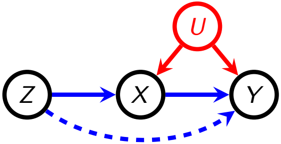
Nested Markov property gives saturated model, so true model of full dimension.

Instrumental Inequalities



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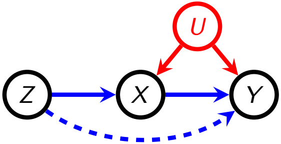

Pearl (1995) showed that if the observed variables are discrete,

$$\max_x \sum_y \max_z P(X = x, Y = y \mid Z = z) \leq 1. \qquad (*)$$

This is the **instrumental inequality**, and can be empirically tested.

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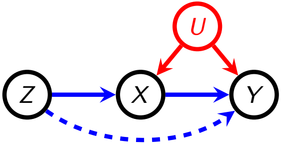

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This is the **instrumental inequality**, and can be empirically tested.

If Z, X, Y are binary, then $(*)$ defines the marginalized DAG model (Bonet, 2001). e.g.

$$P(X = x, Y = 0 \mid Z = 0) + P(X = x, Y = 1 \mid Z = 1) \leq 1$$